Darboux Points and Integrability Analysis of Hamiltonian Systems with Homogeneous Rational Potentials

Michał Studziński
Institute for Theoretical Physics and Astrophysics,
University of Gdańsk
Wita Stwosza 57, PL-80-952 Gdańsk, Poland
and
National Quantum Information Centre of Gdańsk
Wł. Andersa 27, PL-81-824 Sopot, Poland
e-mail: studzinski.m.g@gmail.com
and
Maria Przybylska
Institute of Physics, University of Zielona Góra,
Licealna 9, PL-65–417, Zielona Góra, Poland
e-mail: M.Przybylska@proton.if.uz.zgora.pl

March 4, 2013

Abstract

We study the integrability in the Liouville sense of natural Hamiltonian systems with a homogeneous rational potential $V(\boldsymbol{q})$. Strong necessary conditions for the integrability of such systems were obtained by an analysis of differential Galois group of variational equations along certain particular solutions. These conditions have the form of arithmetic restrictions putted on eigenvalues of Hessian $V''(\boldsymbol{d})$ calculated at a non-zero solution $\boldsymbol{d}$ of equation $\operatorname{grad} V(\boldsymbol{d}) = \boldsymbol{d}$. Such solutions are called proper Darboux points.

It was recently proved that for generic polynomial homogeneous potentials there exist universal relations between eigenvalues of Hessians of the potential taken at all proper Darboux points. The question about the existence of such relations for rational potentials seems to be hard. One of the reason of this fact is the presence of indeterminacy points of the potential and its gradient. Nevertheless, for two degrees of freedom we prove that such relation exists. This result is important because it allows to show that the set of admissible values for eigenvalues of Hessian at a proper Darboux point for potentials satisfying necessary conditions for the integrability is finite. In turn, it gives a tool for classification of integrable rational potentials.

Also, quite recently, it was shoved that for polynomial homogeneous potentials additional necessary conditions for the integrability can be deduced from the existence of improper Darboux points, i.e., points $\boldsymbol{d}$ which are non-zero solution of equation $\operatorname{grad} V(\boldsymbol{d}) = \mathbf{0}$. These new conditions have also the form of arithmetic restrictions imposed on eigenvalues of $V''(\boldsymbol{d})$. In this paper we prove that for rational potentials improper Darboux points give the same necessary conditions for the integrability.
Key words:. Non-integrability criteria; Integrability; Hamiltonian systems; Differential Galois theory; Application of residue calculus

2010 Mathematics Subject Classification: 37J30; 70H05; 70H07; 32A27

1 Introduction and main results

In this paper we consider Hamiltonian systems given by Hamilton's function of the form

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \tag{1.1}$$

where $\boldsymbol{q} = (q_1, \ldots, q_n)$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ are canonical coordinates and momenta in the complex phase space $\mathbb{C}^{2n}$. We assume that potential $V = V(\boldsymbol{q}) \in \mathbb{C}(\boldsymbol{q})$ is a homogeneous rational function. We can write it the following form

$$V(\boldsymbol{q}) := \frac{W(\boldsymbol{q})}{U(\boldsymbol{q})}, \tag{1.2}$$

where $W(\boldsymbol{q})$ and $U(\boldsymbol{q})$ are relatively prime homogeneous polynomials of degrees $r, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, respectively. Hence, potential $V(\boldsymbol{q})$ is a homogeneous function of degree $k := r - s \in \mathbb{Z}$.

Our aim is to investigate the integrability of such systems. Recently, this problem was considered in papers [8, 9] for systems with polynomial homogeneous potentials. Methods developed in these papers are strong. Thanks to them it was possible to give necessary and sufficient conditions for the integrability of homogeneous polynomial potentials of small degree k, and small number of freedom n. The main purpose of this paper is to extend methods and results of the cited papers to systems with rational potentials. Some first attempts for such systematic analysis are contained in B.Sc. thesis and M.Sc. thesis of the first author [16, 17].

The equations generated by Hamiltonian (1.1) have the canonical form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q} = \boldsymbol{p}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} = -V'(\boldsymbol{q}), \tag{1.3}$$

where $V'(\boldsymbol{q}) := \operatorname{grad} V(\boldsymbol{q})$. We assume that the time t is complex. We say that potential V is integrable if this system is integrable in the Liouville sense, i.e., equations (1.3) possess

n functionally independent commuting first integrals. It is useful to consider the equivalence classes of potentials as it was made in [9]. Namely, let $\mathrm{PO}(n,\mathbb{C})$ be the complex projective orthogonal subgroup of $\mathrm{GL}(n,\mathbb{C})$, i.e.,

$$\mathrm{PO}(n,\mathbb{C}) = \{\boldsymbol{A} \in \mathrm{GL}(n,\mathbb{C}),\ |\ \boldsymbol{A}\boldsymbol{A}^T = \alpha \boldsymbol{E}_n,\ \alpha \in \mathbb{C}^\star\}, \tag{1.4}$$

where $\boldsymbol{E}_n$ is n-dimensional identity matrix. We say that V and $\widetilde{V}$ are equivalent if there exists $\boldsymbol{A} \in \mathrm{PO}(n,\mathbb{C})$ such that $\widetilde{V}(\boldsymbol{q}) = V_{\boldsymbol{A}}(\boldsymbol{q}) := V(\boldsymbol{A}\boldsymbol{q})$. Obviously, if V is integrable, then also $V_{\boldsymbol{A}}$ is integrable, for any $A \in \mathrm{PO}(n,\mathbb{C})$. In further considerations a potential means a class of equivalent potentials in the above sense.

The crucial role in our consideration plays the notion of Darboux point. A non-zero $\boldsymbol{d} \in \mathbb{C}^n$ is called a Darboux point of potential V iff $V'(\boldsymbol{d})$ is parallel to $\boldsymbol{d}$. Thus, $\boldsymbol{d}$ is a solution of the following system

$$V'(\boldsymbol{d}) = \gamma \boldsymbol{d}, \tag{1.5}$$

where $\gamma \in \mathbb{C}$, or, equivalently, it satisfies

$$\boldsymbol{d} \wedge V'(\boldsymbol{d}) = 0. \tag{1.6}$$

We say that a Darboux point $\boldsymbol{d}$ is a proper Darboux point iff $V'(\boldsymbol{d}) \neq \boldsymbol{0}$, otherwise it is called an improper one.

Proper Darboux point defines a straight-line particular solution of the system (1.3) given by

$$(\boldsymbol{p}(t), \boldsymbol{q}(t)) = (\varphi(t)\boldsymbol{d}, \dot{\varphi}(t)\boldsymbol{d}), \tag{1.7}$$

where $\varphi(t)$ is a scalar function satisfying

$$\ddot{\varphi} = -\gamma \varphi^{k-1}. \tag{1.8}$$

This equation has first integral

$$h(\varphi, \dot{\varphi}) = \frac{1}{2}\dot{\varphi}^2 + \frac{\gamma}{k}\varphi^k. \tag{1.9}$$

Thus, in fact, we have to our disposal a family of particular solutions parametrised by the value of first integral h.

Let $\boldsymbol{\Gamma}_\varepsilon$ denote the phase curve of particular solution (1.7) for which $h(\varphi, \dot{\varphi}) = \varepsilon$, i.e.,

$$\boldsymbol{\Gamma}_\varepsilon := \left\{ (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n} \ |\ (\boldsymbol{q}, \boldsymbol{p}) = (\varphi \boldsymbol{d}, \dot{\varphi}\boldsymbol{d}),\ h(\varphi, \dot{\varphi}) = \varepsilon \right\}. \tag{1.10}$$

The variational equation along solution (1.7) can be written in the following form

$$\ddot{\boldsymbol{x}} = \varphi(t)^{k-2} V''(\boldsymbol{d})\boldsymbol{x}, \tag{1.11}$$

where $V''(\boldsymbol{d})$ is the Hessian of the potential V calculated at a Darboux point $\boldsymbol{d}$. If matrix $V''(\boldsymbol{d})$ is diagonalisable, then without loss of generality we can assume that the above system splits into a direct sum of second order equations

$$\ddot{x}_i = \varphi(t)^{k-2} x_i, \qquad i = 1, \ldots, n. \tag{1.12}$$

It appears that the analysis of properties of these variational equations gives very strong and computable obstructions to the integrability of potential $V(\boldsymbol{q})$. In the case of proper Darboux point these obstructions were found by J. J. Morales-Ruiz and J.-P. Ramis in [12]. Here we formulate them in the following form.

Theorem 1.1. *Assume that the Hamiltonian system defined by Hamiltonian* (1.3) *with a homogeneous potential $V \in \mathbb{C}(\boldsymbol{q})$ of degree $k \in \mathbb{Z} \setminus \{-2, 0, 2\}$ satisfies the following conditions:*

1. *the potential has a proper Darboux point $\boldsymbol{d}$ satisfying*

$$V'(\boldsymbol{d}) = \gamma \boldsymbol{d}, \quad \text{with} \quad \gamma \in \mathbb{C}^{\star} := \mathbb{C} \setminus \{0\},$$

2. *matrix $\gamma^{-1}V''(\boldsymbol{d})$ is diagonalisable with eigenvalues $\lambda_1, \ldots, \lambda_n$;*
3. *the system is integrable in the Liouville sense with first integrals which are meromorphic in a connected neighbourhood U of phase curve $\boldsymbol{\Gamma}_\varepsilon$ with $\varepsilon \neq 0$, and independent on $U \setminus \boldsymbol{\Gamma}_\varepsilon$.*

Then each pair (k, λ_i) for $i = 1, \ldots, n$ belongs to an item of the following list

$$\begin{array}{llll}
1. & \left(k, p + \dfrac{k}{2}p(p-1)\right), & 2. & \left(k, \dfrac{1}{2}\left[\dfrac{k-1}{k} + p(p+1)k\right]\right), \\
3. & \left(3, -\dfrac{1}{24} + \dfrac{1}{6}(1+3p)^2\right), & 4. & \left(3, -\dfrac{1}{24} + \dfrac{3}{32}(1+4p)^2\right), \\
5. & \left(3, -\dfrac{1}{24} + \dfrac{3}{50}(1+5p)^2\right), & 6. & \left(3, -\dfrac{1}{24} + \dfrac{3}{50}(2+5p)^2\right), \\
7. & \left(4, -\dfrac{1}{8} + \dfrac{2}{9}(1+3p)^2\right), & 8. & \left(5, -\dfrac{9}{40} + \dfrac{5}{18}(1+3p)^2\right), \\
9. & \left(5, -\dfrac{9}{40} + \dfrac{1}{10}(2+5p)^2\right), & 10. & \left(-3, \dfrac{25}{24} - \dfrac{1}{6}(1+3p)^2\right), \\
11. & \left(-3, \dfrac{25}{24} - \dfrac{3}{32}(1+4p)^2\right), & 12. & \left(-3, \dfrac{25}{24} - \dfrac{3}{50}(1+5p)^2\right), \\
13. & \left(-3, \dfrac{25}{24} - \dfrac{3}{50}(2+5p)^2\right), & 14. & \left(-4, \dfrac{9}{8} - \dfrac{2}{9}(1+3p)^2\right), \\
15. & \left(-5, \dfrac{49}{40} - \dfrac{5}{18}(1+3p)^2\right), & 16. & \left(-5, \dfrac{49}{40} - \dfrac{1}{10}(2+5p)^2\right)
\end{array} \tag{1.13}$$

where p is an integer.

The proof of the above theorem is based on an analysis of the differential Galois group of variational equations (1.12). It appears that the existence of n independent first integrals which are in involution puts strong restrictions on this group. Namely, its identity component must be Abelian, see [11]. This condition is translated into arithmetic restrictions on eigenvalues of the Hessian $V''(\boldsymbol{d})$.

REMARK 1.2 *The crucial role in the proof of the above theorem plays the following fact. Assuming that* $k \neq 0$ *and* $\varepsilon \neq 0$, *we make transformation*

$$t \longmapsto z := \frac{1}{k\varepsilon}\varphi(t)^k. \tag{1.14}$$

Then, i-th equation (1.12) *reads*

$$z(1-z)x'' + \left(\frac{k-1}{k} - \frac{3k-2}{2k}z\right)x' + \frac{\lambda_i}{2k}x = 0, \tag{1.15}$$

where prime denotes the differentiation with respect to z. It is exactly the Gauss hypergeometric equation

$$z(1-z)x'' + [c - (a+b+1)z]x' - abx = 0, \tag{1.16}$$

with parameters

$$a+b = \frac{k-2}{2k}, \qquad ab = -\frac{\lambda_i}{2k}, \qquad c = 1 - \frac{1}{k}.. \tag{1.17}$$

For this equation the differential Galois group is perfectly known, see, e.g., [6]. We called transformation (1.14) *the Yoshida transformation as it was found by Haruo Yoshida in [18].*

For the excluded values of $k = \pm 2$ the variational equations do not give any obstruction to the integrability.

Case $k = 0$ is special because for this value of k the phase curves corresponding to a proper Darboux point have form different from that described by (1.10). The necessary integrability conditions for this case were found in [2]. They are following.

Theorem 1.3. *Assume that potential* $V \in \mathbb{C}(\boldsymbol{q})$ *homogeneous of degree* $k = 0$, *and that the following conditions are satisfied:*

1. *there exists a non-zero* $\boldsymbol{d} \in \mathbb{C}^n$ *such that* $V'(\boldsymbol{d}) = \gamma\boldsymbol{d}$ *for a certain* $\gamma \in \mathbb{C}^\star$; *and*
2. *the potential is integrable in the Liouville sense with rational first integrals.*

Then:

1. *all eigenvalues of* $\gamma^{-1}V''(\boldsymbol{d})$ *are integers; and*
2. *the matrix* $V''(\boldsymbol{d})$ *is diagonalizable.*

The fact that $\boldsymbol{d}$ is a Darboux point and V is a homogeneous function of degree k implies that $\boldsymbol{d}$ is an eigenvector of matrix $\gamma^{-1}V''(\boldsymbol{d})$ with the corresponding eigenvalue $\lambda_n = k - 1$. This eigenvalue belongs to the first item of table (1.13) and so it does not give any restriction to the integrability. Later we will called it the trivial eigenvalue of $V''(\boldsymbol{d})$.

For a fixed $k \in \mathbb{Z} \setminus \{-2, 0, 2\}$, the subset of rational numbers $\lambda \in \mathbb{Q}$ such that the pair (k, λ) belongs to an item of the Morales-Ramis table is denoted by $\mathcal{M}_k$. By Theorem 1.3, we have also $\mathcal{M}_0 := \mathbb{Z}$. For the later use we define also the set

$$\mathcal{I}_k = \{\Lambda \in \mathbb{Q} \mid \Lambda + 1 \in \mathcal{M}_k\}. \tag{1.18}$$

Note that for an arbitrary $k \in \mathbb{Z} \setminus \{-2, 0, 2\}$ sets $\mathcal{M}_k$ and $\mathcal{I}_k$ are not finite.

REMARK 1.4 *As it was explained in [4] the assumption that $V''(\boldsymbol{d})$ is diagonalisable is irrelevant. In a case when $V''(\boldsymbol{d})$ is not diagonalisable the necessary conditions for the integrability are exactly the same: if the potential is integrable, then each eigenvalue λ of $V''(\boldsymbol{d})$ belongs to an item of table* (1.13). *Furthermore, the presence of a Jordan block of dimensions greater than 1 gives additional integrability obstructions. Namely, if the Jordan form of $V''(\boldsymbol{d})$ has a block of dimension greater that two then the system is not integrable. Moreover, if it has a two dimensional block and the corresponding λ belongs to the first item of table* (1.13)*, then the system is also not integrable. This facts were proved in [4].*

Necessary conditions for the integrability can be deduced also from an analysis of the differential Galois groups of variational equations along a particular solution given by an improper Darboux point. For polynomial potentials it was done in [13]. Here we generalise this result to the class of rational potentials.

Theorem 1.5. *Assume that homogeneous potential $V \in \mathbb{C}(\boldsymbol{q})$ of degree $k \in \mathbb{Z} \setminus \{-2, 0, 2\}$ possesses an improper Darboux point $\boldsymbol{d}$. If V is integrable in Liouville sense with rational first integrals, then all eigenvalues of $V''(\boldsymbol{d})$ vanish.*

In order to find additional integrability obstructions we restrict our considerations to potentials with two degrees of freedom. In this case for a proper Darboux point $\boldsymbol{d}$ there is only one non-trivial eigenvalue of Hessian $V''(\boldsymbol{d})$. Its value is given by

$$\lambda = \gamma^{-1} \operatorname{Tr} V''(\boldsymbol{d}) - (k-1).$$

We say that $\boldsymbol{d}$ is a simple Darboux point iff $\lambda \neq 1$. Later we will give more algebraic definition of simplicity.

For an improper Darboux point $\boldsymbol{d}$ we have also only one non-trivial eigenvalue of the form

$$\lambda = \operatorname{Tr} V''(\boldsymbol{d}).$$

The trivial eigenvalue of $V''(\boldsymbol{d})$ for an improper Darboux point $\boldsymbol{d}$ is $\lambda_2 = 0$.

In further part of this paper we show that for a homogeneous rational potential of degree $k = r - s$, if number of Darboux points is finite, then there exists $l \leq r + s$ proper Darboux points. For each proper Darboux point $\boldsymbol{d}_i$ we calculate the "shifted" non-trivial eigenvalue

$$\Lambda_i = \lambda_i - 1 = \gamma^{-1} \operatorname{Tr} V''(\boldsymbol{d}_i) - k, \tag{1.19}$$

and we set $\boldsymbol{\Lambda} = (\Lambda_i, \ldots, \Lambda_l) \in \mathbb{C}^l$. It appears that $\boldsymbol{\Lambda}$ cannot be an arbitrary element of $\mathbb{C}^l$. Namely, there is a certain universal relation between them.

Theorem 1.6. *Assume that a rational homogeneous potential $V(q_1, q_2)$ of degree $k \in \mathbb{Z}$ satisfies three conditions:*

C1 *it has $0 < l \leq r + s$ proper Darboux points and all of them are simple;*

C2 *U is not factorisable neither by $(q_2 + \mathrm{i} q_1)$, nor by $(q_2 - \mathrm{i} q_1)$;*

C3 *if W is factorisable by $(q_2 + \mathrm{i}q_1)$, or by $(q_2 - \mathrm{i}q_1)$ then multiplicity of these factor is one.*

Then the shifted non-trivial eigenvalues Λ_i given by (1.19) *satisfy the relation*

$$\sum_{i=1}^{l} \frac{1}{\Lambda_i} = -1. \tag{1.20}$$

In order to formulate a generalisation of the above theorem we have to introduce some definitions. As we will explain later a non-constant rational homogeneous function $F(q_1, q_2)$ can be represented uniquely, up to multiplicative constant, in the following form

$$F(q_1, q_2) = \prod_{i=1}^{r} l_i(q_1, q_2)^{e_i},$$

where $r \in \mathbb{N}$, $e_i \in \mathbb{Z}^\star$ and $l_i(q_1, q_2)$ are homogeneous polynomials of degree 1. We say that $l_i(q_1, q_2)$ with $1 \leq i \leq r$ is a factor of $F(q_1, q_2)$ iff $e_i > 0$. In this case integer e_i is called the multiplicity of factor $l_i(q_1, q_2)$. Let $r_\pm$ and $s_\pm$ be the respective multiplicities of linear factors $(q_2 \pm \mathrm{i}q_1)$ of W and U, respectively. We also define the following step function

$$\theta_{x,y} := \begin{cases} 0 & \text{for} \quad x < y, \\ 1 & \text{for} \quad x \geq y, \end{cases} \tag{1.21}$$

where $x, y \in \mathbb{R}$.

Theorem 1.7. *Assume that a rational homogeneous potential $V(q_1, q_2)$ of degree $k = r - s \in \mathbb{Z}$ satisfies three conditions:*

C1 *it has $0 < l \leq r + s$ proper Darboux points and all of them are simple;*

C2 *neither r_+, nor r_- is equal to $k/2$;*

C3 *neither s_+, nor s_- is equal to $-k/2$.*

Then the shifted non-trivial eigenvalues given in (1.19) *satisfy the relation*

$$\sum_{i=1}^{l} \frac{1}{\Lambda_i} = -1 - \theta_{r_+,2}\frac{r_+}{k - 2r_+} - \theta_{r_-,2}\frac{r_-}{k - 2r_-} + \theta_{s_+,1}\frac{s_+}{k + 2s_+} + \theta_{s_-,1}\frac{s_-}{k + 2s_-}. \tag{1.22}$$

The importance of relations (1.20) and (1.22) follows from the following two theorems.

Theorem 1.8. *Let us consider* (1.20) *or* (1.22) *as an equation for $(\Lambda_1, \ldots, \Lambda_l) \in \underbrace{\mathbb{C}^\star \times \cdots \times \mathbb{C}^\star}_{l}$. Then, for $k \in \mathbb{Z} \setminus \{-2, 2\}$, it has at most a finite number of solutions contained in $\underbrace{\mathcal{I}_k \times \cdots \times \mathcal{I}_k}_{l}$.*

In other words, this theorems says that if the potential of degree $k \in \mathbb{Z} \setminus \{-2, 0, 2\}$ is integrable, then we have only a finite number of choices for eigenvalues of $V''(\boldsymbol{d})$.

The plan of this paper is the following. In Section 2 we describe Darboux points of rational potentials as elements of complex projective space $\mathbb{CP}^{n-1}$ using homogeneous as well as affine coordinates and we show some their general properties. In Section 3 we prove Theorem 1.5. The next Section 4 contains more precise description of properties of Darboux points for homogeneous rational potentials with two degrees of freedom. In particular, in Subsection 4.1 we determine the number of possible proper Darboux points, characterise potentials with multiple proper and improper Darboux points. In the end of this subsection we determine potentials with infinitely many proper Darboux points as well as without proper Darboux points. In Subsection 4.2 we prove Theorems 1.6 and 1.7 which give us universal relations between all eigenvalues of Hessian $V''(\boldsymbol{d})$. The importance of such relations is presented in Subsection 4.3, where the proof of Theorem 1.8 is presented. This theorem is the bases of the classification programme of integrable rational potentials.

In Section 5 we give some applications of the presented theory. In Subsection 5.1 we introduced a class of potentials with a negative degree of homogeneity which satisfy all necessary integrability. We obtain it by reconstruction from the fixed non-trivial eigenvalues calculated at proper Darboux points assuming that their number is maximal and all these eigenvalues are the same and belong to first family from table 1.13. In Subsection 5.2 we distinguish special sub-class of these potentials which have additional first integrals. In the end of this paper in Subsection 5.3 we made analogical analysis like in Subsection 5.1 but for homogeneous rational potential of positive degree of homogeneity.

2 General properties of Darboux points of rational potentials

Here we will use some standard notions of algebraic geometry. Thus, for a collection of polynomials $F_1, \ldots, F_s \in \mathbb{C}[\boldsymbol{q}]$, we denote by $\mathcal{V}(F_1, \ldots, F_s)$ the set of their common zeros, i.e.,

$$\mathcal{V}(F_1, \ldots, F_s) = \{\boldsymbol{q} \in \mathbb{C}^n \mid F_1(\boldsymbol{q}) = \cdots = F_s(\boldsymbol{q}) = 0\}.$$

Such a subset of $\mathbb{C}^n$ is called an algebraic set.

Let us note that a rational potential

$$V(\boldsymbol{q}) = \frac{W(\boldsymbol{q})}{U(\boldsymbol{q})} \tag{2.1}$$

cannot be considered as a function which is well defined for all $\boldsymbol{q} \in \mathbb{C}^n$. In fact, if polynomials $W(\boldsymbol{q})$ and $U(\boldsymbol{q})$ are relatively prime and $\deg U(\boldsymbol{q}) > 0$, then $V(\boldsymbol{q})$ is not well defined whenever $U(\boldsymbol{q}) = 0$. At such a point value of $V(\boldsymbol{q})$ is either infinity or is indefinite. The second case occurs at points $\boldsymbol{q} \in \mathbb{C}$ such that $U(\boldsymbol{q}) = W(\boldsymbol{q}) = 0$. The set of all such points of potential V we denote by $\mathcal{N}(V)$. Thus $\mathcal{N}(V) = \mathcal{V}(U, W)$. The set of poles

$\mathcal{P}(V)$ of potential V is the set of points $\boldsymbol{q} \in \mathbb{C}$ at which the values of V are infinite. Thus $\mathcal{P}(V) = \mathcal{V}(U) \setminus \mathcal{N}(V)$.

We show that for $n = 2$ we have $\mathcal{N}(V) = \{0\}$. In fact, we know that every homogeneous complex polynomial in two variables is a product of linear forms. Thus we can write

$$W(q_1, q_2) = \prod_{i=1}^{r} \left(\alpha_1^{(i)} q_1 + \alpha_2^{(i)} q_2\right), \quad |\alpha_1^{(i)}|^2 + |\alpha_2^{(i)}|^2 \neq 0, \quad \text{for} \quad i = 1, \ldots, r,$$
$$U(q_1, q_2) = \prod_{j=1}^{s} \left(\beta_1^{(j)} q_1 + \beta_2^{(j)} q_2\right), \quad |\beta_1^{(j)}|^2 + |\beta_2^{(j)}|^2 \neq 0, \quad \text{for} \quad j = 1, \ldots, s.$$

If $W(q_1, q_2) = U(q_1, q_2) = 0$ for a certain (q_1, q_2), then

$$\left(\alpha_1^{(i)} q_1 + \alpha_2^{(i)} q_2\right) = \left(\beta_1^{(j)} q_1 + \beta_2^{(j)} q_2\right) = 0,$$

for a certain $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$. If $(q_1, q_2) \neq (0, 0)$, then

$$\beta_1^{(j)} = \gamma \alpha_1^{(i)}, \quad \text{and} \quad \beta_2^{(j)} = \gamma \alpha_2^{(i)},$$

for a certain $\gamma \in \mathbb{C}$. However, it means that U and W have a common factor $\left(\alpha_1^{(i)} q_1 + \alpha_2^{(i)} q_2\right)$ and this is a contradiction with our assumption that U and W are relatively prime.

In order to give a precise definition of Darboux points we consider rational vector $V'(\boldsymbol{q}) \in \mathbb{C}(\boldsymbol{q})^n$. Although we assumed that $V(\boldsymbol{q}) = W(\boldsymbol{q})/U(\boldsymbol{q})$ with relatively prime polynomials W and U, it does not imply that polynomials U and

$$S_i = \frac{\partial W}{\partial q_i} U - W \frac{\partial U}{\partial q_i}, \qquad i = 1, \ldots, n, \tag{2.2}$$

are relatively prime. In other words writing

$$\partial_i V = \frac{\partial V}{\partial q_i} = \frac{S_i}{U^2},$$

we cannot be sure that the sets of indefiniteness of $\partial_i V$ is $\mathcal{V}(S_i, U)$.

From definition (1.6) it follows that if $\boldsymbol{d}$ is a Darboux point of potential V, then $\alpha \boldsymbol{d}$, where $\alpha \in \mathbb{C}^\star$, is also a Darboux point which gives the same phase curves (1.10). This is why it is convenient to consider Darboux points as points in projective space. Thus we consider a non-zero $\boldsymbol{q} \in \mathbb{C}^n$ as the homogeneous coordinates of a point in $\mathbb{CP}^{n-1}$ and we write $[\boldsymbol{q}] := [q_1 : \cdots : q_n] \in \mathbb{CP}^{n-1}$.

In order to avoid formalities we say that $[\boldsymbol{d}] \in \mathbb{CP}^{n-1}$ is a Darboux point of the potential V iff

1. components of $V'(\boldsymbol{q})$ are well defined at $\boldsymbol{q} = \boldsymbol{d}$ and ;

2. $V(\boldsymbol{q})$ is well defined at $\boldsymbol{q} = \boldsymbol{d}$ and;

3. $\boldsymbol{d} \wedge V'(\boldsymbol{d}) = 0$.

The set of all Darboux points of potential V we denote by $\mathcal{D}(V)$. A Darboux point $[\boldsymbol{d}] \in \mathcal{D}(V)$ is called a proper Darboux point iff $V'(\boldsymbol{d}) \neq 0$, otherwise it is called an improper Darboux point. The set of all proper Darboux points of potential V is denoted by $\mathcal{D}^{\star}(V)$. We say that $[\boldsymbol{d}] \in \mathcal{D}(V)$ is an isotropic Darboux point iff

$$d_1^2 + \cdots + d_n^2 = 0.$$

The set of all isotropic Darboux points of potential V is denoted by $\mathcal{D}_0(V)$.

As it was shown in [13] for a polynomial potential V set $\mathcal{D}(V)$ is an algebraic subset of $\mathbb{CP}^{n-1}$. For a rational V it is generally not the case. However, we can consider the following homogeneous polynomials

$$R_{i,j} = q_i \left(\frac{\partial W}{\partial q_j} U - \frac{\partial U}{\partial q_j} W \right) - q_j \left(\frac{\partial W}{\partial q_i} U - \frac{\partial U}{\partial q_i} W \right), \quad 1 \leq i < j \leq n. \tag{2.3}$$

Then the algebraic set

$$\mathcal{R}(V) = \mathcal{V}(R_{1,2}, \ldots, R_{n-1,n}) \subset \mathbb{CP}^{n-1}, \tag{2.4}$$

contains all Darboux points i.e. $\mathcal{D}(V) \subset \mathcal{R}(V)$, and clearly $\mathcal{N}(V) \subset \mathcal{R}(V)$.

It appears that it is convenient to perform some calculations in affine coordinates on $\mathbb{CP}^{n-1}$. They are introduced in the following way. A point in the $(n-1)$ dimensional complex projective space $\mathbb{CP}^{n-1}$ is specified by its homogeneous coordinates $[\boldsymbol{q}] = [q_1 : \cdots : q_n]$, where $\boldsymbol{q} = (q_1, \ldots, q_n) \in \mathbb{C}^n$. We define n open subsets U_i of $\mathbb{CP}^{n-1}$

$$U_i := \Big\{ [q_1 : \cdots : q_n] \in \mathbb{CP}^{n-1} \mid q_i \neq 0 \Big\} \quad \text{for} \quad i = 1, \ldots, n. \tag{2.5}$$

Clearly

$$\mathbb{CP}^{n-1} = \bigcup_{i=1}^{n} U_i, \tag{2.6}$$

and we have natural coordinate maps

$$\theta_i : \mathbb{CP}^{n-1} \supset U_i \to \mathbb{C}^{n-1}, \qquad \theta_i([\boldsymbol{q}]) = (x_1, \ldots, x_{n-1}),$$

where

$$(x_1, \ldots, x_{n-1}) = \left(\frac{q_1}{q_i}, \ldots, \frac{q_{i-1}}{q_i}, \frac{q_{i+1}}{q_i}, \ldots, \frac{q_n}{q_i} \right). \tag{2.7}$$

Each U_i is homeomorphic to $\mathbb{C}^{n-1}$. It is easy to check that charts (U_i, θ_i), $i = 1, \ldots, n$ form an atlas which makes $\mathbb{CP}^{n-1}$ an holomorphic $(n-1)$-dimensional manifold. It is customary to choose one U_i, e.g., U_1, and call it the affine part of $\mathbb{CP}^{n-1}$ and coordinates on them we call affine coordinates.

It means that on $U_1 := \left\{ [\boldsymbol{q}] \in \mathbb{CP}^{n-1} \mid \; q_1 \neq 0 \right\}$ we define affine coordinates

$$\theta_1 : U_1 \to \mathbb{C}^{n-1}, \quad \boldsymbol{x} := (x_1, \dots, x_{n-1}) = \theta_1([\boldsymbol{q}]), \tag{2.8}$$

by

$$x_i = \frac{q_{i+1}}{q_1}, \quad \text{for} \quad i = 1, \dots, n-1. \tag{2.9}$$

For a homogeneous polynomial $F \in \mathbb{C}[\boldsymbol{q}]$ we define its dehomogenisation $f \in \mathbb{C}[x_1, \dots, x_{n-1}]$ in the following way

$$f(x_1, \dots, x_{n-1}) := F(1, x_1, \dots, x_{n-1}).$$

Now we prove the following lemma.

Lemma 2.1.

$$\theta_1(\mathcal{R}(V) \cap U_1) = \mathcal{V}(g_1, \dots, g_{n-1}), \tag{2.10}$$

where polynomials $g_1, \dots, g_{n-1} \in \mathbb{C}[\boldsymbol{x}]$ are given by

$$g_0 := kuw - \sum_{i=1}^{n-1} x_i \left(u\frac{\partial w}{\partial x_i} - w\frac{\partial u}{\partial x_i} \right), \qquad k = r - s, \tag{2.11}$$

and

$$g_i := u\frac{\partial w}{\partial x_i} - w\frac{\partial v}{\partial x_i} - x_i g_0, \quad \textit{for} \quad i = 1, \dots, n-1. \tag{2.12}$$

Moreover, if $[\boldsymbol{d}] \in \mathcal{R}(V) \cap U_1$ is an improper Darboux, then its affine coordinates $\boldsymbol{a} := \theta_1([\boldsymbol{d}]) \in \mathbb{C}^{n-1}$ satisfy $g_0(\boldsymbol{a}) = 0$.

Proof. Darboux points are determined by common zeros of polynomials R_{ij}. We calculate their dehomogenisations r_{ij}.

At first we express partial derivatives of W and U in affine coordinates

$$\begin{aligned}
&\frac{\partial W}{\partial q_1}(\boldsymbol{q}) = q_1^{r-1}\left[rw - \sum_{j=1}^{n-1} x_j \frac{\partial w}{\partial x_j}(\boldsymbol{x}) \right], \qquad \frac{\partial W}{\partial q_i}(\boldsymbol{q}) = q_1^{r-1}\frac{\partial w}{\partial x_{i-1}}(\boldsymbol{x}),\\
&\frac{\partial U}{\partial q_1}(\boldsymbol{q}) = q_1^{s-1}\left[su - \sum_{j=1}^{n-1} x_j \frac{\partial u}{\partial x_j}(\boldsymbol{x}) \right], \qquad \frac{\partial U}{\partial q_i}(\boldsymbol{q}) = q_1^{s-1}\frac{\partial u}{\partial x_{i-1}}(\boldsymbol{x}),
\end{aligned} \tag{2.13}$$

for $i = 2, \dots, n-1$. As

$$r_{i,j}(x_1, \dots, x_{n-1}) := R_{i,j}(1, x_1, \dots, x_{n-1}), \quad \text{for} \quad 1 \leq i < j \leq n-1,$$

through direct calculations we obtain

$$\begin{aligned}
g_i := r_{1,i} = {} & \frac{\partial u(\boldsymbol{x})}{\partial x_{i-1}} w(\boldsymbol{x}) - \frac{\partial w(\boldsymbol{x})}{\partial x_{i-1}} u(\boldsymbol{x}) + \\
& x_{i-1}\left[kuw + \sum_{j=1}^{n-1} x_j \left(w(\boldsymbol{x})\frac{\partial u(\boldsymbol{x})}{\partial x_j} - u(\boldsymbol{x})\frac{\partial w(\boldsymbol{x})}{\partial x_j} \right) \right]
\end{aligned} \tag{2.14}$$

for $1 \le i \le n-1$. Then we easily get

$$r_{i+1,j+1} = x_i g_j - x_j g_i, \qquad \text{for} \quad 1 \le i < j \le n-1. \tag{2.15}$$

Thus clearly $\mathcal{V}(r_{1,2}, \ldots, r_{n-1,n}) = \mathcal{V}(g_1, \ldots, g_{n-1})$.

At an improper Darboux point $[\boldsymbol{d}]$ we have

$$V'(\boldsymbol{d}) = \frac{1}{U(\boldsymbol{d})^2}\left(W'(\boldsymbol{d})U(\boldsymbol{d}) - U'(\boldsymbol{d})W(\boldsymbol{d})\right) = 0.$$

By assumptions, $U(\boldsymbol{d}) \neq 0$, so

$$W'(\boldsymbol{d})U(\boldsymbol{d}) - U'(\boldsymbol{d})W(\boldsymbol{d}) = 0.$$

Passing into affine coordinates we obtain

$$\begin{aligned}
&\frac{\partial W}{\partial q_1}U - \frac{\partial U}{\partial q_1}W = q_1^{r+s-1} g_0(\boldsymbol{x}),\\
&\frac{\partial W}{\partial q_j}U - \frac{\partial U}{\partial q_j}W = q_1^{r+s-1}\left(\frac{\partial w}{\partial x_{j-1}}u - \frac{\partial u}{\partial x_{j-1}}w\right) = q_1^{r+s-1}(g_{j-1}(\boldsymbol{x}) + x_{j-1}g_0(\boldsymbol{x})),
\end{aligned}$$

for $j = 2, \ldots, n$. Thus, in fact at improper Darboux point we have $g_0(\boldsymbol{x}) = 0$, and this ends the proof. $\square$

3 Proof of Theorem 1.5

The variational equations along particular solution

$$t \mapsto (\varphi(t)\boldsymbol{d}, \dot{\varphi}(t)\boldsymbol{d})$$

have the form

$$\ddot{\boldsymbol{x}} = -\varphi(t)^{k-2}V''(\boldsymbol{d})\boldsymbol{x}. \tag{3.1}$$

We make a linear change of variables $\boldsymbol{x} = \boldsymbol{C}\boldsymbol{\eta}$ which transforms (3.1) into

$$\ddot{\boldsymbol{\eta}} = -\varphi(t)^{k-2}\boldsymbol{D}\boldsymbol{\eta}, \qquad \boldsymbol{D} = \boldsymbol{C}^{-1}V''(\boldsymbol{d})\boldsymbol{C}. \tag{3.2}$$

Hence we can choose $\boldsymbol{C}$ in such a way that $\boldsymbol{D}$ is the Jordan form of $V''(\boldsymbol{d})$. System (3.2) contains as a subsystem the following direct product of equations

$$\ddot{\eta}_i = -\lambda_i \varphi(t)^{k-2}\eta_i, \qquad i = 1, \ldots, m \le n,$$

where $\lambda_1, \ldots, \lambda_m$ are pairwise different eigenvalues of $V''(\boldsymbol{d})$.

We have to prove that if $\lambda_i \neq 0$ for a certain $1 \le i \le m$, then the identity component of differential Galois group of system (3.2) is not Abelian. To this end it is enough to show that the identity component of differential Galois group of a single equation

$$\ddot{\eta}_i = -\lambda_i \varphi(t)^{k-2}\eta_i, \quad \text{with} \quad \lambda_i \neq 0, \tag{3.3}$$

is not Abelian for an appropriate choice of φ. For an improper Darboux point $\varphi(t)$ satisfies $\ddot{\varphi} = 0$. So, we can take $\varphi(t) = At$, where $A \in \mathbb{C}$ and we choose A such that $-\lambda_i(At)^{k-2} = t^{k-2}$.

Denoting $\eta := \eta_i$ and $\alpha = k - 2$ we rewrite (3.3) as

$$\ddot{\eta} = t^{\alpha}\eta. \tag{3.4}$$

Now we prove the following lemma

Lemma 3.1. *For $\alpha \in \mathbb{Z} \setminus \{-4, -2, 0\}$ the differential Galois group of equation* (3.4) *is* $\mathrm{SL}(2, \mathbb{C})$.

Proof. For $\alpha \in \mathbb{N}$ the statement of lemma is proved in [13], see proof of Theorem 2.4 in [13]. Hence, we assume that $\alpha = -\beta \le 0$. Let us introduce new independent variable $\tau = 1/t$. Equation (3.4) transforms into

$$\eta'' + \frac{2}{\tau}\eta' - s^{\beta-4}\eta = 0,$$

where prime denotes the differentiation with respect to τ. Now, introducing new dependent variable $\xi = \tau\eta$ we obtain

$$\xi'' = \tau^{\beta-4}\xi. \tag{3.5}$$

The transformation that we made, does not change the differential Galois group of the considered equation. Hence, our lemma is already proved for $\beta > 4$, that is, for $\alpha < -4$. We have only to prove our lemma for $\alpha = -1$, and for $\alpha = -3$.

We prove this with the help of the Kovacic algorithm, for details, see [7]. This algorithm allows to decide whether the considered second order linear equation with rational coefficients has only Liouvillian solution. At the same time it provides detailed information about the differential Galois group of this equation. In particular, if equation does not admit a Liouvillian solution, then its differential Galois group is $\mathrm{SL}(2, \mathbb{C})$.

We apply the Kovacic algorithm in its original formulation [7]. All definitions and notation is exactly the same as in this paper.

The algorithm is divided into mutually exclusive cases. Only in first three cases the equation admits a Liouvillian solution. If none of these three cases occurs, then the differential Galois group of the considered equation is $\mathrm{SL}(2, \mathbb{C})$. In this case the whole group coincides with its identity component, so both are not Abelian.

We show that for $\alpha = -1$ none of first three cases of the Kovacic algorithm is allowed. To this end we apply Theorem on page 8 of [7] which gives the necessary conditions for the respective cases. Equation (3.4) has two singularities: a pole of order $-\alpha$ at $t = 0$ and the infinity which has also order $-\alpha$.

The necessary condition for case I is that the order of infinity is either even or greater than 2. For $\alpha = -1$ it is not the case.

Case II cannot occur because the necessary condition for it is that there is at least one pole of odd order greater than 2 or else of order two.

Case III cannot also occurs because the necessary condition for it is that the infinity has order at least 2.

As result, we showed that for $\alpha = -1$ equation (3.4) has no Liouvillian solution and its differential Galois group is $\mathrm{SL}(2, \mathbb{C})$.

Using the same theorem for $\alpha = -3$ we easily notice that if equation (3.4) has a Liouvillian solution, then we fall into case II of the algorithm. In this case exponents at the singularities are

$$E_0 = \{3\} \qquad \text{and} \qquad E_\infty = \{0, 2, 4\},$$

According to the algorithm if this case occurs, then there exist $e_0 \in E_0$ and $e_\infty \in E_\infty$ such that number

$$d = \frac{1}{2}(e_\infty - e_0)$$

is a non-negative integer. But it is impossible because e_0 is odd and e_∞ even. Thus also for $\alpha = -3$ differential Galois group of (3.4) is $\mathrm{SL}(2, \mathbb{C})$. □

Let us notice that if $\alpha = -4$, then the degree of homogeneity of the potential $k = -2$. In this case equation (3.4) has two Liouvillian solutions

$$\eta_1 = te^{\frac{\sqrt{\lambda}}{t}}, \qquad \eta_2 = te^{-\frac{\sqrt{\lambda}}{t}}$$

satisfying the following relations

$$\frac{\eta_1'}{\eta_1} = \frac{t - \sqrt{\lambda}}{t^2} \in \mathbb{C}(t), \qquad \frac{\eta_2'}{\eta_2} = \frac{t + \sqrt{\lambda}}{t^2} \in \mathbb{C}(t), \qquad \eta_1\eta_2 = t^2 \in \mathbb{C}(t).$$

For $\alpha = -2$, i.e., $k = 0$, the variational equation has solutions

$$\eta_1 = t^{\frac{1}{2}+\sqrt{1+4\lambda}}, \qquad \eta_2 = t^{\frac{1}{2}-\sqrt{1+4\lambda}},$$

satisfying the following relations

$$\frac{\eta_1'}{\eta_1} = \frac{1 + \sqrt{1 + 4\lambda}}{2t} \in \mathbb{C}(t), \qquad \frac{\eta_2'}{\eta_2} = \frac{1 - \sqrt{1 + 4\lambda}}{2t} \in \mathbb{C}(t), \qquad \eta_1\eta_2 = t \in \mathbb{C}(t).$$

The above shows that in cases with $k = -2$ and $k = 0$ the differential Galois of variational equations is Abelian as it is subgroup of diagonal group. It is an open problem if the presence of a Jordan block gives additional obstructions for the integrability in these cases. For $\alpha = 0$, corresponding to $k = 2$, variational equation (3.4) becomes an equation with constant coefficient, thus it does not give any integrability obstructions.

4 Two degrees of freedom

4.1 More about properties of Darboux points

In this section we assume that $n = 2$. A rational homogeneous potential can be written in the following form

$$\begin{gathered} V(\boldsymbol{q}) = \frac{W(\boldsymbol{q})}{U(\boldsymbol{q})}, \quad \text{where} \\ W(\boldsymbol{q}) = \sum_{i=0}^{r} w_{r-i} q_1^{r-i} q_2^{i}, \quad U(\boldsymbol{q}) = \sum_{j=0}^{s} u_{s-j} q_1^{s-j} q_2^{j}. \end{gathered} \tag{4.1}$$

We assume that polynomials W and U are relatively prime. Since polynomials W and U are homogeneous of degrees r and s, respectively, potential V is a homogeneous function of degree $k = r - s$.

We denote by $z = q_2/q_1$ the affine coordinate on $U_1 \subset \mathbb{CP}^1$. Then, dehomogenizations $w(z)$, $u(z)$ and $v(z)$ of polynomials $W(\boldsymbol{q})$, $U(\boldsymbol{q})$ and potential $V(\boldsymbol{q})$ have the following forms

$$w(z) = \sum_{i=0}^{r} w_{r-i} z^i, \qquad u(z) = \sum_{j=0}^{s} u_{s-j} z^j, \qquad v(z) = \frac{w(z)}{u(z)}.$$

On $U_2 \subset \mathbb{CP}^1$ as a coordinate we take $\zeta = 1/z$. Now, dehomogenizations of U, V and W are following

$$\widetilde{u}(\zeta) := U(\zeta, 1) = \zeta^s u(1/\zeta) = \sum_{i=1}^{s} u_i \zeta^i, \qquad \widetilde{v}(\zeta) := \zeta^k v(1/\zeta), \qquad \widetilde{w}(\zeta) := \zeta^r w(1/\zeta). \tag{4.2}$$

We can also write

$$w(z) = \gamma \prod_{i=1}^{l} (z - a_i)^{\alpha_i}, \quad \text{and} \quad u(z) = \prod_{i-1}^{m} (z - b_i)^{\beta_i}, \tag{4.3}$$

where $\alpha_i, \beta_i \in \mathbb{N}$. We assume that for $w(z) = \gamma$ for $l = 0$, and $u(z) = 1$ for $m = 0$.

For $n = 2$ Darboux points of potential V are zeros of one rational function

$$F(\boldsymbol{q}) = \boldsymbol{q} \wedge V'(\boldsymbol{q}) = q_1 \frac{\partial V}{\partial q_2} - q_2 \frac{\partial V}{\partial q_1}.$$

Its dehomogenisation $f(z)$ is following

$$f(z) = F(1, z) = (1 + z^2) v'(z) - kzv(z). \tag{4.4}$$

Now, polynomials $g := g_1$ and $h := g_0$, which are defined in Lemma 2.1, have the forms

$$\begin{gathered} h(z) = kw(z)u(z) - z\left[w'(z)u(z) - u'(z)w(z)\right], \\ g(z) = (1 + z^2)\left[w'(z)u(z) - u'(z)w(z)\right] - kzw(z)u(z). \end{gathered} \tag{4.5}$$

Notice that $g = fu^2$.

For an investigation of equivalent potentials in affine coordinates we introduce the following notation. Let $A = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$, with $a^2 + b^2 \neq 0$ be an element of $\mathrm{PO}(2, \mathbb{C})$, and $R(\boldsymbol{q})$ a rational homogeneous function of degree p. We define $R_A(\boldsymbol{q}) := R(A\boldsymbol{q})$. Let $r(z)$ and $\widetilde{r}(\zeta)$ be dehomogenizations of $R(\boldsymbol{q})$. Then the dehomogenizations $r_A(z)$ and $\widetilde{r}_A(\zeta)$ are given by

$$r_A(z) = (a + bz)^p\, r(\tau_A(z)), \qquad \widetilde{r}_A(\zeta) = (a - b\zeta)^p\, r(\tau_A^{-1}(\zeta)), \tag{4.6}$$

where

$$\tau_A(z) = \frac{az - b}{bz + a}, \qquad \tau_A^{-1}(z) = \frac{az + b}{-bz + a}$$

In order to avoid numerous repetitions we assume implicitly in this section that polynomials U and W are relatively prime. Moreover, in order to simplify formulations instead of saying that $z \in \mathbb{C}$ is an affine coordinate of a Darboux point we just say that z is a Darboux point. Taking into account this convention the sets of all Darboux points, proper and improper Darboux points of potential V which are contained in the affine part of $\mathbb{CP}^1$ can be defined in the following way

$$\begin{aligned} \mathcal{D}(V) &:= \{z \in \mathbb{C} \mid g(z) = 0 \quad \text{and} \quad u(z) \neq 0\}\,, \\ \mathcal{D}^\star(V) &:= \{z \in \mathbb{C} \mid g(z) = 0 \text{ and } h(z) \neq 0 \text{ and } u(z) \neq 0\}\,, \\ \mathcal{D}(V) \setminus \mathcal{D}^\star(V) &:= \{z \in \mathbb{C} \mid g(z) = 0 \text{ and } h(z) = 0 \text{ and } u(z) \neq 0\}\,, \end{aligned} \tag{4.7}$$

respectively. We also define the set of isotropic Darboux points like

$$\mathcal{D}_0(V) = \{z \in \{-\mathrm{i}, \mathrm{i}\} \mid g(z) = 0 \text{ and } u(z) \neq 0\}\,. \tag{4.8}$$

If polynomial g is not identically zero, then the number of Darboux points of the potential is bounded by its degree. Just direct calculations show the following.

Proposition 4.1. *The degree of polynomial g is not greater than $r + s$. Moreover,* $\deg g = r + s$ *if and only if*

$$w_1 u_0 - w_0 u_1 \neq 0. \tag{4.9}$$

It is worth to notice that condition

$$w_1 u_0 - w_0 u_1 = 0, \tag{4.10}$$

means that polynomial

$$S_1 = \frac{\partial W}{\partial q_1} U - W \frac{\partial U}{\partial q_1},$$

vanishes at points $(0, q_2)$ with $q_2 \in \mathbb{C}^\star$. Thus, if $U(0, 1) \neq 0$, then $[0 : 1]$ is a Darboux point of V.

We show that only radial potentials have infinite number of Darboux points.

Proposition 4.2. *A homogeneous rational potential $V \in \mathbb{C}(q_1, q_2)$ possesses an infinite number of Darboux points if and only if V is radial, i.e., it has the following form*

$$V(\boldsymbol{q}) := c(q_1^2 + q_2^2)^l, \qquad k = 2l, \tag{4.11}$$

for a certain $l \in \mathbb{Z}$.

Proof. If potential has the form (4.11), then

$$V'(\boldsymbol{q}) = \frac{k}{r^2} V(\boldsymbol{q})\boldsymbol{q}, \qquad r^2 := q_1^2 + q_2^2,$$

so for any $\boldsymbol{q} \in \mathbb{C}^2$, we have

$$\boldsymbol{q} \wedge V'(\boldsymbol{q}) = 0.$$

Hence, in fact, an arbitrary non-zero $\boldsymbol{q} \in \mathbb{C}^2$ gives a Darboux point.

On the other hand, if V has an infinite number Darboux points, then rational function $f(z)$ given by (4.4) has infinite number of zeros, so it vanishes identically. Thus, we have

$$(1 + z^2)v'(z) - kzv(z) = 0. \tag{4.12}$$

Using separation of variables we obtain the following solution of this equation

$$v(z) = c(1 + z^2)^{k/2}.$$

Because, by assumptions, $v(z)$ is rational, $k = 2l$ for a certain integer l. The homogenisation of $v(z)$ gives (4.11), and this finishes the proof. $\square$

We show that if the considered potential is not radial, then we can always choose its representative such that the polynomial $g(z)$ has degree $r + s$.

Proposition 4.3. *Assume that the potential has a finite number of Darboux points. Then it has a representative such that the corresponding polynomial $g(z)$ has degree $r + s$.*

Proof. From formulae (4.2) it follows that

$$u_0 = \widetilde{u}(0), \qquad u_1 = \widetilde{u}\,'(0), \quad \text{and} \quad w_0 = \widetilde{w}(0), \qquad w_1 = \widetilde{w}\,'(0) \tag{4.13}$$

So, the condition that $g(z)$ has degree $r + s$ can be written in the following form

$$t := \widetilde{w}(0)\widetilde{u}\,'(0) - \widetilde{w}\,'(0)\widetilde{u}(0) \neq 0. \tag{4.14}$$

Let us take a potential $V_A(\boldsymbol{q}) := V(A\boldsymbol{q})$ equivalent to $V(\boldsymbol{q})$. We assume that $A = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$, with $a^2 + b^2 \neq 0$.

Let us assume the V is not radial and for an arbitrary $A \in \mathrm{PO}(2, \mathbb{C})$, the polynomial $g_A(z)$ corresponding to V_A has degree smaller that $r + s$. Since $v_A(z) = w_A(z)/u_A(z)$, it implies that

$$t_A := \widetilde{w}_A(0)\widetilde{u}\,'_A(0) - \widetilde{w}'_A(0)\widetilde{u}_A(0) = 0, \tag{4.15}$$

for all $A \in \mathrm{PO}(2,\mathbb{C})$. We have

$$\widetilde{u}_A(\zeta) = (a - b\zeta)^s\, u(\tau_A^{-1}(\zeta)), \qquad \widetilde{w}_A(\zeta) = (a - b\zeta)^r\, w(\tau_A^{-1}(\zeta)), \tag{4.16}$$

and simple calculations give

$$\widetilde{u}_A(0) = a^s u(x), \qquad \widetilde{u}'_A(0) = a^s \left[(1 + x^2)u'(x) - sxu(x)\right], \tag{4.17}$$

and

$$\widetilde{w}_A(0) = a^r w(x), \qquad \widetilde{w}'_A(0) = a^r \left[(1 + x^2)w'(x) - rxw(x)\right], \tag{4.18}$$

where $x := a/b \in \mathbb{C}$. Now, condition (4.15) reads

$$(1 + x^2)(w'(x)u(x) - w(x)u'(x)) - (r - s)xw(x)u(x) = 0 \quad \text{for} \quad x \in \mathbb{C}. \tag{4.19}$$

Dividing by $u(x)^2$ we obtain differential equation

$$(1 + x^2)v'(x) - kv(x) = 0, \qquad v(x) = \frac{w(x)}{u(x)}, \quad k = r - s,$$

coinciding with equation (4.12). But from the proof of Proposition 4.2 we know that for odd k this equation does not have a non-zero rational solutions, and for even k, the only rational solutions are radial. In any case we have a contradiction and this finishes the proof. $\square$

The number of different Darboux points of the potential, if it is finite, can be smaller, then maximal predicted by Proposition 4.1. It can happens in the following cases.

1. Potential V has an improper Darboux point;
2. Polynomial U has a multiple linear factor;
3. Polynomial $U(q_1, q_2)$ has a linear factor $(\pm \mathrm{i}q_1 + q_2)$;
4. Potential V has a multiple proper Darboux point.

It appears that cases 1–3 occurs only if polynomials g and h have a common root. Here it is also worth to remark that an improper Darboux point appears in a non-generic situation. In fact, if $z_\star$ is an improper Darboux point, then it is a root of polynomial g and h, so they are not relatively prime.

Proposition 4.4. *Assume that $k \neq 0$. Then polynomials $g(z)$ and $h(z)$ possess a common root $z_\star$ if and only if $z_\star$ is a multiple root of either $w(z)$, or $u(z)$.*

Proof. If $z_\star$ is a multiple root of $w(z)$, then $w(z_\star) = w'(z_\star) = 0$ and formulae (4.5) give immediately $g(z_\star) = h(z_\star) = 0$. Similarly, a multiple root of $u(z)$ means $u(z_\star) = u'(z_\star) = 0$ and gives immediately $g(z_\star) = h(z_\star) = 0$.

Now we assume that $z_\star$ is a common root of $g(z)$ and $h(z)$, i.e. $g(z_\star) = h(z_\star) = 0$. From formulae (4.5) it follows that

$$zh(z) + g(z) = w'(z)u(z) - u'(z)w(z).$$

Thus for $z = z_\star$ we obtain

$$w'(z_\star)u(z_\star) - u'(z_\star)w(z_\star) = 0. \tag{4.20}$$

But then from (4.5) $0 = g(z_\star) = -kz_\star w(z_\star)u(z_\star)$ and $0 = h(z_\star) = kw(z_\star)u(z_\star)$. For $k \neq 0$ we obtain that $w(z_\star) = 0$ or $u(z_\star) = 0$. If we choose $w(z_\star) = 0$, then (4.20) simplifies to $w'(z_\star)u(z_\star) = 0$, that gives $w'(z_\star) = 0$ because u and w are relatively prime. It means that $z_\star$ is a multiple root of $w(z)$. If we choose $u(z_\star) = 0$, then (4.20) simplifies to $u'(z_\star)w(z_\star) = 0$, that gives $u'(z_\star) = 0$ because u and w are relatively prime. As result $z_\star$ is a multiple root of $u(z)$. □

In our further considerations we need to know multiplicities of common roots of g and u, as well as, g and w. The answer to this question is given in the following two lemmas.

Lemma 4.5. *Assume that* $\operatorname{mult}(w, z_\star) = l \geq 1$. *Then*

$$\operatorname{mult}(g, z_\star) = \begin{cases} > l & \text{if} \quad z_\star \in \{-\mathrm{i}, \mathrm{i}\}, \quad \text{and} \quad k = 2l, \\ = l & \text{if} \quad z_\star \in \{-\mathrm{i}, \mathrm{i}\}, \quad \text{and} \quad k \neq 2l, \\ = l - 1 & \text{otherwise.} \end{cases} \tag{4.21}$$

Proof. We can write

$$w = (z - z_\star)^l \widetilde{w} \qquad l, \in \mathbb{N},$$

where $\widetilde{w}$ is a polynomial such that $\widetilde{w}(z_\star) \neq 0$. Inserting the above expression into the definition of g we obtain

$$g(z) = (z - z_\star)^{l-1}\widetilde{g}(z), \tag{4.22}$$

where

$$\widetilde{g} = \left[l(z^2+1) - kz(z - z_\star)\right] \widetilde{w}u + (z^2+1)(z - z_\star)(\widetilde{w}'u - u'\widetilde{w}). \tag{4.23}$$

Hence

$$\widetilde{g}(z_\star) = l(z_\star^2 + 1)\widetilde{w}(z_\star)u(z_\star). \tag{4.24}$$

Now, if $z_\star \notin \{-\mathrm{i}, \mathrm{i}\}$, then $\widetilde{g}(z_\star) \neq 0$ because $l \neq 0$, $\widetilde{w}(z_\star)u(z_\star) \neq 0$, and $z_\star^2 + 1 \neq 0$. Hence, in this case $\operatorname{mult}(g, z_\star) = l - 1$.

Now, assume that $z_\star = \mathrm{i}$. Then we obtain

$$g(z) = (z - z_\star)^{l-1}\widetilde{g}(z), \tag{4.25}$$

where

$$\widetilde{g} = [l(z + \mathrm{i}) - kz]\, \widetilde{w}u + (z^2+1)(\widetilde{w}'u - u'\widetilde{w}), \tag{4.26}$$

and thus

$$g(\mathrm{i}) = \mathrm{i}l(2l-k)\widetilde{w}(\mathrm{i})u(\mathrm{i}).$$

From the above formula, it follows directly that if $k \neq 2l$, then $\operatorname{mult}(g,\mathrm{i}) = l$, and otherwise $\operatorname{mult}(g,\mathrm{i}) > l$. Similar calculations for $z_\star = -\mathrm{i}$ give that if $k \neq 2l$, then $\operatorname{mult}(g,-\mathrm{i}) = l$, and otherwise $\operatorname{mult}(g,-\mathrm{i}) > l$. In this way we conclude our proof. $\square$

Notice that polynomials u and w enter symmetrically in the definition (4.5) of g. Hence we have also the following statement.

Lemma 4.6. *Assume that* $\operatorname{mult}(u,z_\star) = l \geq 1$. *Then*

$$\operatorname{mult}(g,z_\star) = \begin{cases} > l & \text{if} \quad z_\star \in \{-\mathrm{i},\mathrm{i}\}, \quad \text{and} \quad k = 2l, \\ = l & \text{if} \quad z_\star \in \{-\mathrm{i},\mathrm{i}\}, \quad \text{and} \quad k \neq 2l, \\ = l-1 & \text{otherwise.} \end{cases} \tag{4.27}$$

In our considerations we have to know multiplicities of common roots of polynomials $g(z)$ and $h(z)$.

Lemma 4.7. *Assume that $z_\star$ is a common root of polynomials $g(z)$ and $h(z)$, and*

$$l := \max\left\{\operatorname{mult}(w,z_\star), \operatorname{mult}(u,z_\star)\right\}.$$

Then

1. *if* $z_\star = 0$, *then* $\operatorname{mult}(h,z_\star) > \operatorname{mult}(g,z_\star) = l-1$;
2. *if* $z_\star \notin \{-\mathrm{i},0,\mathrm{i}\}$, *then* $\operatorname{mult}(h,z_\star) = \operatorname{mult}(g,z_\star) = l-1$;
3. *if* $z_\star \in \{-\mathrm{i},\mathrm{i}\}$, *and* $l \neq k/2$, *then* $\operatorname{mult}(g,z_\star) = \operatorname{mult}(h,z_\star) + 1 = l$;
4. *if* $z_\star \in \{-\mathrm{i},\mathrm{i}\}$, *and* $l = k/2$, *then* $\operatorname{mult}(g,z_\star) > \operatorname{mult}(h,z_\star) + 1 = l$.

Proof. By Proposition 4.4, $z_\star$ is either a multiple root of $u(z)$, or a multiple root of $w(z)$. Let us assume that it is a multiple root of $w(z)$. We can write

$$w(z) = (z-z_\star)^l\widetilde{w}(z), \qquad \widetilde{w}(z_\star) \neq 0,$$

where $\widetilde{w}(z)$ is a polynomial. Then, from definition of $h(z)$, see (4.5), we obtain

$$h(z) = (z-z_\star)^{l-1}\widetilde{h}(z),$$

where

$$\widetilde{h}(z) = k(z-z_\star)u(z)\widetilde{w}(z) - z\left[lu(z)\widetilde{w}(z) + (z-z_\star)\left(u'(z)\widetilde{w}(z) + u(z)\widetilde{w}'(z)\right)\right].$$

Hence

$$\widetilde{h}(z_\star) = -lz_\star u(z_\star)\widetilde{w}(z_\star).$$

As $l > 0$, $u(z_\star) \neq 0$, and $\widetilde{w}(z_\star) \neq 0$, we have that for $z_\star \neq 0$, $\operatorname{mult}(h,z_\star) = l-1$, and $\operatorname{mult}(h,0) > l-1$. Now, combining this with Lemma 4.5 we obtain the desired results. In the case when $z_\star$ is a multiple root of polynomial $u(z)$ the proof is similar. $\square$

Now we want to distinguish those potentials which do not admit any proper Darboux point.

Lemma 4.8. *Let $V = W/U$ with relatively prime polynomials $W, U \in \mathbb{C}[\boldsymbol{q}]$ of respective degrees r and s be a homogeneous rational potential of degree $k = r - s \neq 0$. If V does not have any proper Darboux point, then it is equivalent to the potential*

$$V(\boldsymbol{q}) = c(q_1 + \mathrm{i}q_2)^{\alpha}(q_1 - \mathrm{i}q_2)^{\beta}, \quad \alpha + \beta = k, \quad \alpha, \beta \in \mathbb{Z}, \qquad c \in \mathbb{C}^{\star}. \tag{4.28}$$

except for the case when $k = 2l > 0$, and either W, or U, has a factor $(q_1 \pm \mathrm{i}q_2)$ with multiplicity l.

Proof. From definitions (4.8) it follows that if V has no proper Darboux points, then all roots of polynomial g are either roots of polynomial u, or polynomial h. If a root of g is also a root of h, then by Proposition 4.4, this common root is either a root of polynomial u, or a root of polynomial w. In effect, all roots of g are either roots u, or roots of w.

We notice, that the potential V has only a finite number of Darboux points. In fact, for otherwise it is radial and has a proper Darboux point. Thanks to this fact, by Proposition 4.3, we can assume that degree of polynomial g is $r + s$. Polynomials u and w have the form (4.3) with all $\alpha_i > 1$ and all $\beta_i > 1$. We can write polynomial g in the following form

$$g(z) = \delta \prod_{i=1}^{l'} (z - a_i)^{\alpha'_i} \prod_{i=1}^{m'} (z - b_i)^{\beta'_i} \tag{4.29}$$

with $l' \leq l$, $m' \leq m$. From assumptions it follows that for $k = 2l > 0$, if $a_j \in \{-\mathrm{i}, \mathrm{i}\}$ for a certain $1 \leq j \leq l'$, then $\alpha_j \neq l$. This is why, by Lemma 4.5, $\alpha'_j \leq \alpha_j$, for $1 \leq j \leq l'$. By the same reason $\beta'_j \leq \beta_j$ for $1 \leq j \leq m'$.

Now, let us assume that $l' + m' > 2$. Then, either there exists $1 \leq j \leq l'$ such that $\alpha'_j = \alpha_j - 1$, or there exists $1 \leq j \leq m'$ such that $\beta'_j = \beta_j - 1$. By Proposition 4.3, we can assume that $\deg g = r + s$, thus from (4.29) we obtain

$$r + s = \sum_{i=1}^{l'} \alpha'_i + \sum_{i=1}^{m'} \beta'_i \leq \sum_{i=1}^{l'} \alpha_i + \sum_{i=1}^{m'} \beta_i - 1 < \sum_{i=1}^{l} \alpha_i + \sum_{i=1}^{m} \beta_i = r + s. \tag{4.30}$$

But this is impossible. This is why $l' + m' \leq 2$.

If $l' + m' < l + m$, then

$$r + s = \sum_{i=1}^{l'} \alpha'_i + \sum_{i=1}^{m'} \beta'_i < \sum_{i=1}^{l} \alpha_i + \sum_{i=1}^{m} \beta_i = r + s.$$

But it is impossible. So, in effect, $l' + m' = l + m$ and it is possible only when $l' = l$ and $m' = m$.

If for a certain $1 \leq i \leq l$, $a_i \notin \{-\mathrm{i}, \mathrm{i}\}$, then $\alpha'_i = \alpha_i - 1$, and

$$r + s = \sum_{i=1}^{l} \alpha'_i + \sum_{i=1}^{m} \beta'_i < \sum_{i=1}^{l} \alpha_i + \sum_{i=1}^{m} \beta_i - 1 < r + s.$$

But it is impossible, so $a_i \in \{-\mathrm{i}, \mathrm{i}\}$, for $1 \leq i, \leq l$. In a similar way we show that $b_i \in \{-\mathrm{i}, \mathrm{i}\}$, for $1 \leq i, \leq m$. In effect the potential has the form (4.28), and this finishes our proof. □

Now we characterize potentials with multiple Darboux points. We say that z_* is a multiple proper Darboux point if $g(z_*) = g'(z_*) = 0$, $h(z_*) \neq 0$ and $u(z_*) \neq 0$.

Proposition 4.9. *If a homogeneous rational potential of degree homogeneity $k \in \mathbb{Z}^*$ with two degrees of freedom has a multiple proper non-isotropic Darboux point, then it is equivalent to a potential of the form* (4.1) *with coefficients satisfying the following conditions*

1. $w_{r-1}u_s = u_{s-1}w_r$,
2. $kw_ru_s = 2\left(w_{r-2}u_s - w_ru_{s-2}\right)$,
3. $kw_ru_s \neq 0$.

Proof. Let us calculate $g'(z)$

$$g'(z) = (1+z^2)(w''(z)u(z) - u''(z)w(z)) + (2-k)z(w'(z)u(z) - u'(z)w(z)) - kw(z)u(z). \tag{4.31}$$

From above-mentioned considerations we know that at a multiple proper Darboux point we have $g(z_\star) = g'(z_\star) = 0$ and $h(z_\star) \neq 0$. Because, by assumption, the considered Darboux point is not isotropic we can move it using a rotation into a point with affine coordinate $z_\star = 0$. Using explicit forms of polynomial $w(z)$ and $u(z)$ substituted into conditions $g(0) = g'(0)$, $h(0) \neq 0$ and $u(0) \neq 0$ we obtain the thesis of this proposition. □

We recall that $z_\star$ is a multiple improper Darboux point iff $g(z_\star) = g'(z_\star) = h(z_\star) = 0$ and $u(z_\star) \neq 0$.

Proposition 4.10. *If a homogeneous rational potential of degree homogeneity $k \in \mathbb{Z}^\star$ with two degrees of freedom has a multiple improper non-isotropic Darboux point, then it is equivalent to a potential of the form* (4.1) *with coefficients satisfying the following conditions*

$$w_r = w_{r-1} = w_{r-2} = 0, \quad \textit{and} \quad u_s \neq 0.$$

Proof. As in the previous proposition we can assume without loss of generality that $z_\star = 0$. Then conditions $g(0) = g'(0) = h(0)$ give the following system

$$w_{r-1}u_s = u_{s-1}w_r, \qquad kw_ru_s = 2\left(w_{r-2}u_s - w_ru_{s-2}\right), \qquad kw_ru_s = 0,$$

and taking into account that, by assumption $k \neq 0$ and $u(0) = u_s \neq 0$ the solution of this system gives the thesis of our proposition. □

An interested reader can formulate analogous propositions for potentials with multiple isotropic Darboux points. Without loss and generality one can assume that the affine coordinate of an isotropic Darboux point is $z_\star = \mathrm{i}$.

4.2 Relations between non-trivial eigenvalues at various proper Darboux points

We start this section with calculation of non-trivial eigenvalue of $V''(\boldsymbol{d})$ in terms of affine coordinate z of Darboux point $[\boldsymbol{d}]$. We assume that the considered Darboux point is proper and it satisfies $V'(\boldsymbol{d}) = \gamma\boldsymbol{d}$ with $\gamma \in \mathbb{C}^{\star}$. Hence, our aim is to evaluate function

$$\lambda = \gamma^{-1}\,\mathrm{Tr}\,V''(\boldsymbol{q}) - (k-1) \tag{4.32}$$

at a point $\boldsymbol{q} \neq \boldsymbol{0}$ satisfying $V'(\boldsymbol{q}) = \gamma\boldsymbol{q}$, and express the result by means of the affine coordinate $z = q_2/q_1$.

Since $V(q_1, q_2) = q_1^k v(z)$ we can express partial derivatives of V at point (q_1, q_2) in terms of variables (q_1, z). They have the following forms

$$\begin{aligned} &\frac{\partial V}{\partial q_1} = q_1^{k-1}\left(kv - zv'\right), \qquad \frac{\partial V}{\partial q_2} = q_1^{k-1}v', \\ &\frac{\partial^2 V}{\partial q_1^2} = q_1^{k-2}\left\{(k-1)\left(kv - zv'\right) + z\left(zv'' - (k-1)v'\right)\right\}, \qquad \frac{\partial^2 V}{\partial q_2^2} = q_1^{k-2}v''. \end{aligned} \tag{4.33}$$

As $v = w/u$, the first component of equality $V'(\boldsymbol{q}) = \gamma\boldsymbol{q}$ gives

$$\gamma = q_1^{k-2}\left(kv - zv'\right) = \frac{kwu - z(w'u - wu')}{u^2} = \frac{h}{u^2},$$

so we have immediately that $q_1^{k-2} = \gamma u^2/h$. Then

$$\lambda = \gamma^{-1}\,\mathrm{Tr}\,V''(\boldsymbol{q}) - (k-1) = \gamma^{-1}\left\{q_1^{k-2}[(k-1)(kv - zv')] + q_1^{k-2}[(1+z^2)v'' - (k-1)zv']\right\}$$
$$-(k-1) = \frac{u^2}{h}[(1+z^2)v'' - (k-1)zv'].$$

Taking into account that

$$v' = \frac{w'u - wu'}{u^2}, \quad v'' = \frac{w''u - wu''}{u^2} + \frac{2u'}{u^3}(wu' - w'u)$$

we obtain

$$\lambda = \frac{u^2}{h}\left\{\frac{(1+z^2)(w''u - wu'') - (k-1)z(w'u - wu')}{u^2} - \frac{2u'(1+z^2)(w'u - wu')}{u^3}\right\}.$$

But the above expression is evaluated only at proper Darboux points, so $g = 0$ at these points. Thus, we have $(1+z^2)(w'u - wu') = kzwu$, and the final form is the following

$$\lambda = \frac{1}{h}\left[(1+z^2)(w''u - wu'') - (k-1)zw'u - (k+1)zu'w\right] = \frac{g'}{h} + 1. \tag{4.34}$$

The last equality one can verify by the direct check. We see that if a proper Darboux point is multiple, then $\lambda = 1$.

If the considered Darboux point $z_\star$ is simple, then the inverse of the corresponding shifted eigenvalue $\Lambda_\star = \lambda_\star - 1$ is given by the following formula

$$\frac{1}{\Lambda_\star} = \frac{h(z_\star)}{g'(z_\star)}. \tag{4.35}$$

It is crucial to notice that in the right hand side of the above formula is the residue of rational function $h(z)/g(z)$ at point $z = z_\star$.

4.2.1 Proof of Theorem 1.6

We choose such a representative potential that the polynomial g given by (4.5) has degree $r + s$, see Proposition 4.3 and comments after it. This guarantees that all Darboux points are located in the affine part of the projective line $\mathbb{CP}^1$, and are roots of g. Next, we define a meromorphic differential form ω which in the affine part of $\mathbb{CP}^1$ is given by

$$\omega(z) = \frac{h(z)}{g(z)}\,\mathrm{d}\,z,$$

and, in a neighbourhood of infinity, in local coordinate $\zeta = 1/z$ it has the form

$$\widetilde{\omega}(\zeta) = \omega\left(\frac{1}{\zeta}\right) = -\frac{h(1/\zeta)}{g(1/\zeta)}\frac{\mathrm{d}\,\zeta}{\zeta^2}. \tag{4.36}$$

At first we assume that polynomials g and h are relatively prime and all roots of g are simple. This implies that $l = r + s$ simple roots of g are proper Darboux points. Taking into account the above facts we can calculate residues $\operatorname{res}(\omega, z_i)$ for $i = 1, \ldots, r + s$

$$\operatorname{res}(\omega, z_i) = \frac{h(z_i)}{g'(z_i)} = \frac{1}{\Lambda(z_i)} = \frac{1}{\Lambda_i}. \tag{4.37}$$

Now, we calculate residue at infinity $\operatorname{res}(\omega, \infty) = \operatorname{res}(\widetilde{\omega}, 0)$. Let us write polynomials g and h in the following form

$$g = \sum_{i=0}^{r+s} g_{r+s-i} z^i, \qquad h = \sum_{i=0}^{r+s-1} h_{r+s-1-i} z^i,. \tag{4.38}$$

Using definition (4.5) we find that

$$g_0 = -h_0 = u_1 w_0 - u_0 w_1 \neq 0, \tag{4.39}$$

see Proposition 4.1.

Now, we rewrite (4.36) in the following form

$$\widetilde{\omega}(\zeta) = \frac{A(\zeta)}{\zeta}\mathrm{d}\zeta, \quad \text{where} \quad A(\zeta) := -\frac{h(1/\zeta)}{\zeta g(1/\zeta)} \tag{4.40}$$

We show that $A(0) = 1$. In fact, we can write

$$A(\zeta) = \frac{a(\zeta)}{b(\zeta)}, \qquad a(\zeta) := -\zeta^{r+s-1} h(1/\zeta), \quad b(\zeta) := \zeta^{r+s} g(1/\zeta)$$

Clearly, $a(\zeta)$ and $b(\zeta)$ are polynomials and $a(0) = b(0) = g_0$. Hence, $A(0) = 1$ as we claimed.

$$\operatorname{res}(\omega, \infty) = \operatorname{res}(\widetilde{\omega}, 0) = 1.$$

According to the global residue theorem we have

$$\sum_{i=1}^{r+s} \frac{1}{\Lambda_i} = \sum_{i=1}^{r+s} \operatorname{res}(\omega, z_i) = -\operatorname{res}(\omega, \infty) = -1.$$

In this way we proved Theorem 1.6 for $l = r + s$.

Now, we consider the general case. Let $z_1, \ldots, z_l$ denotes roots of g corresponding to proper Darboux points. By assumption C2, if $g(z_\star) = 0 = u(z_\star)$, then $z_\star \notin \{-\mathrm{i}, \mathrm{i}\}$. Them, since $\operatorname{mult}(g, z_\star) \geq 1$ by Lemma 4.6 and assumption C2, $\operatorname{mult}(u, z_\star) = 1 + \operatorname{mult}(g, z_\star) > 1$. So, $z_\star$ is a multiple root of $u(z)$. Thus by Proposition 4.4, $z_\star$ is a root of $h(z)$. In effect, all roots of $g(z)$ which are not proper Darboux points are also roots of $h(z)$. By assumptions C2, C3, and Lemma 4.7 we know that if $z_\star$ is a common root of g and h, then $\operatorname{mult}(g, z_\star) \leq \operatorname{mult}(h, z_\star)$. Hence, we have $g = f\bar{g}$, and $h = f\bar{h}$, where f, $\bar{g}$ and $\bar{h}$ are polynomials, and $\bar{g}(z_\star) \neq 0$. We denote also $\Lambda_i = \Lambda(z_i)$ for $i = 1, \ldots, l$. Now, the form ω reads

$$\omega(z) = \frac{h(z)}{g(z)}\,\mathrm{d}\,z = \frac{\bar{h}(z)}{\bar{g}(z)}\,\mathrm{d}\,z.$$

It has poles at $z_1, \ldots, z_l$ and at the infinity. Notice that we have

$$\frac{1}{\Lambda_i} = \frac{h(z_i)}{g'(z_i)} = \frac{f(z_i)\bar{h}(z_i)}{f'(z_i)\bar{g}(z_i) + f(z_i)\bar{g}'(z_i)} = \frac{\bar{h}(z_i)}{\bar{g}'(z_i)} = \operatorname{res}(\omega, z_i), \quad i = 1, \ldots, l,$$

where $f(z_i) \neq 0$ and $\bar{g}(z_i) = 0$ for $i = 1, \ldots, l$. Additionally, we have also $\operatorname{res}(\omega, \infty) = 1$, hence by the global residue theorem we obtain (1.20) and this finishes the proof.

4.2.2 Proof of Theorem 1.7

If $s_+ = s_- = 0$, and $r_+, r_- \in \{0, 1\}$, then assumptions of Theorem 1.6 are satisfied. So, in this case our theorem is valid.

Now, we consider general case. We can write

$$w(z) = (z - \mathrm{i})^{r_+}(z + \mathrm{i})^{r_-}\widetilde{w}(z), \qquad u(z) = (z - \mathrm{i})^{s_+}(z + \mathrm{i})^{s_-}\widetilde{u}(z), \tag{4.41}$$

where $\widetilde{w}(z)$ and $\widetilde{u}(z)$ are polynomials, and $\widetilde{w}(\pm\mathrm{i}) \neq 0$, $\widetilde{u}(\pm\mathrm{i}) \neq 0$. As we assumed that $u(z)$ and $w(z)$ are relatively prime, at most one number in pairs (r_+, s_+) and (r_-, s_-) is different from zero.

Calculating function $g(z)$ and $h(z)$ using (4.41) we obtain

$$\begin{aligned} g(z) =&(z-\mathrm{i})^{r_++s_+}(z+\mathrm{i})^{r_-+s_-}\bar{g}(z), \\ h(z) =&(z-\mathrm{i})^{r_++s_+-1}(z+\mathrm{i})^{r_-+s_--1}\bar{h}(z), \end{aligned} \tag{4.42}$$

where

$$\begin{aligned} \bar{g}(z) =&(1+z^2)(\widetilde{w}'(z)\widetilde{u}(z)-\widetilde{u}'(z)\widetilde{w}(z)) \\ &+[(r_+-s_+)(z+\mathrm{i})+(r_--s_-)(z-\mathrm{i})-kz]\,\widetilde{w}(z)\widetilde{u}(z), \\ \bar{h}(z) =& \left\{k(z^2+1)-z\left[(r_+-s_+)(z+\mathrm{i})+(r_--s_-)(z-\mathrm{i})\right]\right\}\widetilde{w}(z)\widetilde{u}(z) \\ &-z(z^2+1)(\widetilde{w}'(z)\widetilde{u}(z)-\widetilde{u}'(z)\widetilde{w}(z)). \end{aligned} \tag{4.43}$$

Thanks to above equations we obtain differential form $\omega(z)$ as follows

$$\omega(z)=\frac{h(z)}{g(z)}\,\mathrm{d}\,z=\frac{\bar{h}(z)}{g_1(z)}\,\mathrm{d}\,z, \quad \text{where} \quad g_1(z)=(1+z^2)\bar{g}(z)$$

and polynomials $\bar{h}(z)$, $\bar{g}(z)$ are relatively prime.

Notice that all finite poles of form $\omega(z)$ are all proper Darboux points z_i, infinity and $\pm\mathrm{i}$. Under assumptions of Theorem 1.7, all poles are simple. Calculations of residues at proper Darboux points as well as at infinity are similar to these in the proof of the previous theorem and we obtain

$$\operatorname{res}(\omega,z_i)=\frac{1}{\Lambda_i}, \qquad \operatorname{res}(\omega,\infty)=1.$$

Residues at $z=\pm\mathrm{i}$ are calculated in the following way

$$\operatorname{res}(\omega,\pm\mathrm{i})=\frac{\bar{h}(\pm\mathrm{i})}{g_1'(\pm\mathrm{i})}.$$

Because

$$g_1'(z)=2z\bar{g}(z)+(z^2+1)\bar{g}'(z),$$

thus we obtain

$$g_1'(\pm\mathrm{i})=\pm2\mathrm{i}\bar{g}(\pm\mathrm{i})$$

but

$$\begin{aligned} &\bar{g}(\mathrm{i})=2\mathrm{i}\left(r_+-s_+-\frac{k}{2}\right)u(\mathrm{i})\widetilde{w}(\mathrm{i}), && \bar{h}(\mathrm{i})=2(r_+-s_+)u(\mathrm{i})\widetilde{w}(\mathrm{i}), \\ &\bar{g}(-\mathrm{i})=-2\mathrm{i}\left(r_--s_--\frac{k}{2}\right)u(-\mathrm{i})\widetilde{w}(-\mathrm{i}), && \bar{h}(-\mathrm{i})=-2(r_--s_-)u(-\mathrm{i})\widetilde{w}(-\mathrm{i}). \end{aligned}$$

Now we finish calculation of the residues

$$\operatorname{res}(\omega,\pm\mathrm{i})=\frac{\bar{h}(\pm\mathrm{i})}{\pm2\mathrm{i}\bar{g}(\pm\mathrm{i})}=\frac{r_\pm-s_\pm}{k-2(r_\pm-s_\pm)}.$$

Applying the global residue theorem we obtain relation (1.22).

4.3 Finiteness of admissible spectra for integrable potentials

At first we recall the notion of a limit point of a set. Let T be a topological space and A a subset of T. A point $x \in T$ is a limit point of A, iff in every open neighbourhood U contains points of A, i.e., $U \cap A \neq \emptyset$.

We start this section with a certain technical lemma. Let $\mathcal{X}$ be a discrete subset of $\mathbb{R}_+ = (0, \infty)$ such that its only limit point is 0. We consider the following equation

$$X_1 + \cdots + X_m = c, \qquad c > 0, \tag{4.44}$$

and we look for its solutions $\boldsymbol{X} = (X_1, \ldots, X_m) \in \underbrace{\mathcal{X} \times \cdots \times \mathcal{X}}_{m} = \mathcal{X}^m$. We prove the following lemma.

Lemma 4.11. *For arbitrary $c > 0$ equation* (4.44) *has at most a finite number of solutions in $\mathcal{X}^m$.*

Proof. We prove this lemma by induction with respect to m. For $m = 1$ the statement of the lemma is evidently true. So, assume that it is true for $m - 1$. We have to show that it is true for m.

First we notice that if $\boldsymbol{X} = (X_1, \ldots, X_m)$ is a solution of equation (4.44), than at least one its component, let us say X_m, is not smaller than c/m. In fact, if we have $X_i < c/m$, for $i = 1, \ldots, m$, then

$$c = X_1 + \ldots + X_m < m\frac{c}{m} = c.$$

This contradiction proves our claim. Hence without loss of the generality we can assume that $X_m \geq c/m$ and we rewrite equation (4.44) in the following form

$$X_1 + \ldots + X_{m-1} = c - X_m. \tag{4.45}$$

Now, it is enough to notice that X_m belongs to the set $\{X \in \mathcal{X} \mid X > c/m\}$ which is finite because $\mathcal{X}$ is discrete and its only limit point is 0. Thus we have only a finite number of possible choices for X_m. For each such choice, equation (4.45), considered as equation with unknown $(X_1, \ldots, X_{m-1})$ has, by the induction assumption, a finite number of solutions. Thus, the number of solution of equation (4.44) is finite. $\square$

We recall that $\mathcal{M}_k$ denotes a subset of rational numbers λ specified by the table in Theorem 1.1 for a given k, e.g., for $|k| > 5$ we have

$$\mathcal{M}_k = \left\{ p + \frac{k}{2}p(p-1) \mid p \in \mathbb{Z} \right\} \cup \left\{ \frac{1}{2}\left[\frac{k-1}{k} + p(p+1)k\right] \mid p \in \mathbb{Z} \right\}. \tag{4.46}$$

We define the following sets

$$\mathcal{I}_k := \{\Lambda \in \mathbb{Q} \mid \lambda = \Lambda + 1 \in \mathcal{M}_k\}, \tag{4.47}$$

and

$$\mathcal{X}_k := \left\{ X \in \mathbb{Q} \mid X = \frac{1}{\Lambda}, \quad \Lambda \in \mathcal{I}_k \right\}. \tag{4.48}$$

In order to describe properties of the above defined sets its convenient to consider them as subsets of compactified real line $\mathbb{R} \cup \{\infty\} \simeq \mathbb{RP}^1$.

Proposition 4.12. *Assume that $k \in \mathbb{N} \setminus \{2\}$. Then*

1. *If $\Lambda \in \mathcal{I}_k$, then $\Lambda \geq -1$.*
2. *The infinity $\{\infty\}$ is the only limit point of set $\mathcal{I}_k$ in $\mathbb{R} \cup \{\infty\}$.*
3. *If $x \in \mathcal{X}_k$, then either $x \leq -1$, or $x > 0$, i.e. $\mathcal{X}_k \subset \mathbb{R} \setminus (-1, 0]$.*
4. *Only $\{0\}$ is a limit point of $\mathcal{X}_k$.*

Proposition 4.13. *Assume that $k \in -\mathbb{N} \setminus \{-2\}$. Then*

1. *If $\Lambda \in \mathcal{I}_k$, then $\Lambda \leq 0$.*
2. *The minus infinity $\{-\infty\}$ is the only limit point of set $\mathcal{I}_k$ in $\mathbb{R} \cup \{-\infty\}$.*
3. *If $x \in \mathcal{X}_k$, then $x < 0$, i.e. $\mathcal{X}_k \subset (-\infty, 0]$.*
4. *Only $\{0\}$ is the limit point of $\mathcal{X}_k$.*

Easy proofs of the above propositions we left to the reader. Now we can pass to the proof of Theorem 1.8.

Proof. At first we assume $k \in \mathbb{N} \setminus \{2\}$. We can write equation (1.22) in the form

$$\sum_{i=1}^{l} X_i = c, \qquad c \in \mathbb{Q}, \tag{4.49}$$

where $X_i := 1/\Lambda_i$, for $i = 1, \ldots, l$. Let N denote the number of solutions of this equation in $\mathcal{X}_k^l$, and N_p for $0 \leq p \leq l$, denote the number of solutions of this equation in $\mathcal{X}_k^l$ such that they have p negative components. We have

$$N = \sum_{p=0}^{l} N_p.$$

We will show that $N_p < \infty$ for $0 \leq p \leq l$. In fact, set $\mathcal{X}_k$ has only a finite number of negative elements. Thus for a given p we have a finite number of choices of p negative elements X_i. Let $X_l, X_{l-1}, \ldots, X_{l-p+1} \in \mathcal{X}_k$ be negative. Then

$$\sum_{i=1}^{l-p} X_i = c' > 0,$$

where $X_1, \ldots, X_{l-p} \in \mathcal{X}_k$ and $X_i > 0$ for $i = 1, \ldots, l-p$. By Lemma 4.11, there is only a finite number of solution of this equation, and this finishes the proof for $k \in \mathbb{N} \setminus \{2\}$.

For $k \in -\mathbb{N} \setminus \{-2\}$ we write equation (1.22) in the form

$$-\sum_{i=1}^{l} X_i = c, \qquad c \in \mathbb{Q}, \tag{4.50}$$

where $X_i := 1/\Lambda_i$, for $i = 1, \dots, l$. Putting $Y_i := -X_i$ for $i = 1, \dots, n$, and $\mathcal{Y}_k := -\mathcal{X}_k$ we have to show that number of solutions of equation

$$\sum_{i=1}^{l} Y_i = c, \qquad c \in \mathbb{Q},$$

such that $(Y_1, \dots, Y_l) \in \mathcal{Y}_k^l$ is finite. By Proposition 4.13, if $y \in \mathcal{Y}_k$, then $y > 0$ and only 0 is a limit point of $\mathcal{Y}_k$. Hence we can proceed like in the case of positive k. □

5 Applications

After determination of all necessary integrability conditions for a given r and s:

- all non-trivial eigenvalues Λ_i of the Hessian must belong to $\mathcal{M}_k$, where $k = r - s$, and
- appropriate relations between these eigenvalues calculated at all proper Darboux points must exist

one can try to classify all integrable rational potentials.

This problem will be considered elsewhere. Here we only show two one-parameter families of potentials satisfying these conditions. Both belong to the special class of rational potentials of the following form

$$V = q_1^{-s} \sum_{i=0}^{r} v_{r-i} q_1^{r-i} q_2^{i}, \quad v_i \in \mathbb{C}. \tag{5.1}$$

In such a case

$$w = v = \sum_{i=0}^{r} v_{r-i} z^{i}, \qquad u = 1, \qquad v = w.$$

The maximal number of proper Darboux points is $r+1$ provided $s \neq 0$. In this section we show two families of such potentials that satisfy all known integrability conditions and between them some integrable cases were found.

5.1 Example with $k \in \mathbb{Z}_- \setminus \{-2\}$

In this section we will consider a family of potentials (5.1) that are homogeneous of degree $k = -\kappa$, where $\kappa > 0$. We assume that they possess maximal number $r+1$ proper Darboux points and the non-trivial eigenvalues are the same at all proper Darboux points

i.e. $\lambda_1 = \ldots = \lambda_{r+1} = \lambda$ and belong to the first item of the Morales-Ramis table. This means that

$$\lambda = p - \frac{1}{2}\kappa p(p-1), \qquad \text{for some} \quad p \in \mathbb{Z}$$

or

$$\Lambda := \lambda - 1 = \frac{1}{2}\left(-\kappa p^2 + (\kappa+2)p - 2\right). \tag{5.2}$$

If we assume that we have $r+1$ simple proper Darboux points different from $\pm\mathrm{i}$ and this number is exactly equal to

$$r + 1 = -\frac{1}{2}\left(-\kappa p^2 + (\kappa+2)p - 2\right), \tag{5.3}$$

then relation

$$\sum_{i=1}^{r+1} \frac{1}{\Lambda} = -1 \tag{5.4}$$

obviously holds. Now we will reconstruct the form of potential starting from these values of spectra of Hessians at all proper Darboux points.

Because the number of proper Darboux points is maximal thus function $g(z)$ defined in (4.5) has only simple roots and can be written as

$$g(z) = \alpha \prod_{i=1}^{r+1}(z - z_i), \qquad \alpha \in \mathbb{C}^\star. \tag{5.5}$$

Using above form of $g(z)$ and decomposition on simple fractions we can calculate $h(z)/g(z)$

$$\frac{h(z)}{g(z)} = \sum_{i=1}^{r+1} \frac{h(z_i)}{g'(z_i)} \frac{1}{z - z_1} = \sum_{i=1}^{r+1} \frac{1}{\Lambda_i} \frac{1}{z - z_i} = \frac{1}{\Lambda} \sum_{i=1}^{r+1} \frac{1}{z - z_i} = \frac{1}{\Lambda} \frac{g'(z)}{g(z)}.$$

Putting equation (5.2) into above formula we find

$$h(z) = \frac{2}{-\kappa p^2 + (\kappa+2)p - 2} g'(z).$$

Finally if we use expressions (4.5) for function $g(z)$ and $h(z)$ we obtain second order linear differential equation

$$(1+z^2)\frac{\mathrm{d}^2 v(z)}{\mathrm{d}\, z} + \left(\kappa + 2 - \frac{\kappa p^2 - (\kappa+2)p}{2}\right) z \frac{\mathrm{d}\, v(z)}{\mathrm{d}\, z} + \left(\kappa - \kappa\frac{\kappa p^2 - (\kappa+2)p}{2}\right) v(z) = 0.$$

Applying transformation in a form $z \mapsto x = \mathrm{i}z$, where $\mathrm{i}^2 = -1$ we obtain

$$(1-x^2)\frac{\mathrm{d}^2 v(x)}{\mathrm{d}\, x^2} - \left(\kappa + 2 - \frac{\kappa p^2 - (\kappa+2)p}{2}\right) x \frac{\mathrm{d}\, v(x)}{\mathrm{d}\, x} + \frac{1}{2}\kappa p(\kappa p - (\kappa+2)) v(x) = 0.$$

If we rewrite this equation as

$$(1-x^2)\frac{\mathrm{d}^2 v(x)}{\mathrm{d}\, x^2} + (\beta - \alpha - (\alpha+\beta+2)x)\frac{\mathrm{d}\, v(x)}{\mathrm{d}\, x} + r(r+\alpha+\beta+1)v(x) = 0 \tag{5.6}$$

where

$$\alpha = \beta = \frac{2(p-1)+\kappa(2+p-p^2)}{4}, \qquad r = \frac{p\,(\kappa(p-1)-2)}{2}, \tag{5.7}$$

then we recognise immediately that this is equation defining Jacobi polynomials $P_r^{(\alpha,\beta)}(x)$ of degree r. Jacobi polynomials can be written as

$$P_r^{(\alpha,\beta)}(x) = 2^{-r}\sum_{i=0}^{r}\binom{r+\alpha}{i}\binom{r+\beta}{r-i}(x-1)^{r-i}(x+1)^i.$$

It means that potential $v(z)$ has the form

$$v(z) = P_r^{(\alpha,\beta)}(\mathrm{i}z) = P_r^{(\alpha,\beta)}\left(\mathrm{i}\frac{q_2}{q_1}\right).$$

and its homogenization gives the final form of our potential

$$V(q_1,q_2) = q_1^{-\kappa}v(\mathrm{i}z) = q_1^{-\kappa}P_r^{(\alpha,\beta)}\left(\mathrm{i}\frac{q_2}{q_1}\right). \tag{5.8}$$

Notice that we obtain two-parameter family of potentials which satisfy all known necessary integrability conditions. The role of parameters play $p \in \mathbb{Z}$ and $\kappa \in \mathbb{N}$, see expressions in (5.7).

5.2 Special class of integrable rational potentials

We can try to fix p and κ and use the direct method in order to find additional first integral. Condition $r > 0$, where r is given by (5.7) gives the following inequality for κ

$$\kappa > \frac{2}{p-1}.$$

For $p = 1$ and small k we made such experiments but without success. The situation changes for $p = 2$ that gives $r + 1 = \kappa - 1$ i.e. $r = \kappa - 2$ for $\kappa > 2$ and $\alpha = \beta = 1/2$. In this case we have

$$P_r^{\left(\frac{1}{2},\frac{1}{2}\right)}(z) = \frac{1}{(r+1)!}\left(\frac{3}{2}\right)_r U_r(z)$$

where $(x)_r$ means the Pochhammer symbol, see e.g. formulae for Jacobi polynomials on Wolfram page [15]. Here U_r is the Chebyshev polynomial of the second kind determined by the following formula

$$U_r(z) = \sum_{i=0}^{\left[\frac{r}{2}\right]}\frac{(-1)^i(r-i)!(2z)^{r-2i}}{i!(r-2i)!}.$$

Substitution $z = \mathrm{i}q_2/q_1$ gives the following form

$$
\begin{aligned}
V(q_1, q_2) &= q_1^{-r-2} P_r^{(\frac{1}{2},\frac{1}{2})}\left(\mathrm{i}\frac{q_2}{q_1}\right) = \frac{C}{q_1^{2r+2}} \sum_{i=0}^{\left[\frac{r}{2}\right]} 2^{-2i} \frac{(r-i)!}{i!(r-2i)!} q_1^{2i} q_2^{r-2i} \\
&= \frac{C}{q_1^{2r+2}\rho}\left[\left(\frac{\rho+q_2}{2}\right)^{r+1} + (-1)^r \left(\frac{\rho-q_2}{2}\right)^{r+1}\right]
\end{aligned}
\tag{5.9}
$$

where $\rho = \sqrt{q_1^2 + q_2^2}$ and

$$
C = \frac{\left(\frac{3}{2}\right)_r (r\mathrm{i})^r}{(r+1)!}.
$$

Such potentials have appeared in [3]. They can be written also as

$$
V_n = \frac{1}{\rho}\left[\left(\frac{\rho+q_2}{2}\right)^{n+1} + (-1)^n \left(\frac{\rho-q_2}{2}\right)^{n+1}\right], \tag{5.10}
$$

for negative $n = -r-2$.

Example 5.1 *The first two non-trivial integrable rational potentials are following*

$$
\begin{aligned}
V_{-3}(q_1, q_2) &= q_1^{-3} P_1^{(\frac{1}{2},\frac{1}{2})}\left(\mathrm{i}\frac{q_2}{q_1}\right) = \frac{q_2}{q_1^4}, \\
V_{-4}(q_1, q_2) &= q_1^{-4} P_2^{(\frac{1}{2},\frac{1}{2})}\left(\mathrm{i}\frac{q_2}{q_1}\right) = \frac{q_1^2 + 4q_2^2}{q_1^6}.
\end{aligned}
$$

with the corresponding first integrals

$$
\begin{aligned}
I_{-3}(q_1, q_2, p_1, p_2) &= p_1(q_2 p_1 - q_1 p_2) + \frac{q_1^2 + 4q_2^2}{2q_1^4}, \\
I_{-4}(q_1, q_2, p_1, p_2) &= p_1(q_2 p_1 - q_1 p_2) + \frac{4q_2(q_1^2 + 2q_2^2)}{q_1^6}.
\end{aligned}
$$

In papers [14, 3] authors shown that potential (5.10) is integrable for all $n \in \mathbb{Z}$. The first integral of the Hamiltonian system with potential (5.10) is the following

$$
I(q_1, q_2, p_1, p_2) = p_1(q_2 p_1 - q_1 p_2) + \frac{1}{2} q_1^2 V_{n-1}.
$$

Failures of the direct search of additional first integrals for other small values of p and κ suggests the non-integrability for $p \neq 2$. Since all known integrability conditions due to variational equations are satisfied the only tool for proving the non-integrability are higher order variational equations. It is known result that if the identity component of differential Galois group of m-th order variational equations for some $m \geq 1$ is non-Abelian,

then Hamiltonian system is not integrable in the Liouville sense [11]. But the dimension of homogeneous linear systems corresponding to higher order variational equations grows very quickly with growing m and determination of its differential Galois group becomes a very hard problem. Thus in practice usually we look for logarithms in solutions of first as well as higher order variational equations. In some cases one can prove that the presence of logarithms in solutions of higher order variational equations of a certain degree $m \geq 1$ implies that their differential Galois group has non-Abelian identity component and as result to prove strictly the non-integrability. This is the case when the first order variational equations are the direct sum of Lamé equations [11, 8, 9] or when linear homogeneous equations are completely reducible [1]. None of these cases holds for Hamiltonian systems governed for potentials of the form (5.8) but one can check for small p and κ that for $p \neq 2$ logarithmic terms in solutions of higher order variational equations appear, in contrary for $p = 2$, there is no such terms.

5.3 Example with $k \in \mathbb{Z}_+ \setminus \{0, 2\}$

In this section we consider homogeneous potentials of a positive degree. Similarly to the previous case we assume that potential possesses maximal number $r+1$ of simple proper Darboux points different from $\pm\mathrm{i}$, and all eigenvalues λ_i for $i = 1, \ldots, r+1$ belong to first item of table (1.13), i.e.,

$$\lambda_i \in \left\{ p + \frac{1}{2}kp(p-1) \mid p \in \mathbb{Z} \right\}.$$

so, we have

$$\Lambda_i := \lambda_i - 1 \in \mathcal{L}_k, \qquad \mathcal{L}_k := \left\{ \frac{1}{2}(p-1)(kp+2) \mid p \in \mathbb{Z} \right\}. \tag{5.11}$$

Set $\mathcal{L}_k$ possesses only one negative element equal to -1, so to satisfy relation (5.4) we must take at least two negative values, for example $\Lambda_r = \Lambda_{r+1} = -1$ and $\Lambda_i > 0$ for $i = 1, \ldots, r-1$. We assume also that $\Lambda_1 = \Lambda_2 = \ldots = \Lambda_{r-1} = \Lambda = (p-1)(kp+2)/2$, then obviously

$$r - 1 = \frac{1}{2}(p-1)(kp+2). \tag{5.12}$$

We can decompose quotient $h(z)/g(z)$ as follows

$$\frac{h(z)}{g(z)} = \frac{1}{\Lambda}\sum_{i=1}^{r-1}\frac{1}{z-z_i} - \frac{1}{z-z_r} - \frac{1}{z-z_{r+1}} = \frac{1}{\Lambda}\sum_{i=1}^{r+1}\frac{1}{z-z_i} - \left(1+\frac{1}{\Lambda}\right)\left(\frac{1}{z-z_r} + \frac{1}{z-z_{r+1}}\right),$$

where affine coordinates of Darboux points z_r and z_{r+1} play role of parameters. Now using (5.5) and (5.11) we obtain

$$\begin{aligned} h(z) &= \frac{1}{\Lambda}g'(z) - \left(1+\frac{1}{\Lambda}\right)\left(\frac{1}{z-z_r} + \frac{1}{z-z_{r+1}}\right)g(z) \\ &= \frac{2}{(p-1)(2+kp)}g'(z) - \frac{p(2+k(p-1))}{(p-1)(2+kp)}\left(\frac{1}{z-z_r} + \frac{1}{z-z_{r+1}}\right)g(z). \end{aligned}$$

From definitions of $g(z)$ and $h(z)$ given in (4.5) we obtain second order differential equation

$$
\begin{aligned}
&v''(z) + p(z)v'(z) + q(z)v(z) = 0,\\
&p(z) = -\frac{p(2-k+kp)}{2(z-a)} - \frac{p(2-k+kp)}{2(z-b)} + \frac{(1+p)(2-2k+kp)z}{2(1+z^2)},\\
&q(z) = \frac{k(2+k(-1+p))p(ab-z^2)}{2(z-a)(z-b)(1+z^2)},
\end{aligned}
\tag{5.13}
$$

where $a = z_r$ and $b = z_{r+1}$. Now we have to find its solutions which are polynomials of degree r. This degree, for given p and k, is defined by equation (5.12). Substitution of the expansion

$$
v(z) = \sum_{i=0}^{r} v_i z^i, \qquad v_i \in \mathbb{C}.
$$

into (5.13) gives a system of homogeneous linear equations of dimension $r+1$ on indefinite coefficients v_i for $i = 0, \ldots, r$. This system reads

$$
\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}, \qquad \boldsymbol{v} = [v_0, \ldots, v_r]^T. \tag{5.14}
$$

Entries of matrix $\boldsymbol{A}$ depend on a and b. This system has a non-zero solution iff all minors of $\boldsymbol{A}$ of degree $r+1$ vanish. These conditions have the form of non-linear polynomial equations for a and b. Among all solutions of these equations we choose those which give as solution $v(z)$ a polynomial of degree r. Then we calculate $V(q_1, q_2) = q_1^k v(q_2/q_1)$. In order to obtain a rational potential which is not a polynomial one, the number $s = r - k$ must be greater than zero. This gives the following condition on k and p

$$
s = p + \frac{k}{2}(p-2)(1+p) > 0.
$$

Thus, let us start our analysis for small k, and small $|p|$. For $k = 1$ and $p \in \{-2, -1, 0, 1\}$ there is no any non-trivial solution of (5.14). For $p = 2$, and $p = -3$, system (5.14) possesses non-trivial solutions with $r + 1 = 4$ proper Darboux points. But all obtained potentials possess some multiple proper Darboux points or $a = \mathrm{i}$ or $b = \mathrm{i}$. For $p = 3$ one can find the following potential satisfying all assumptions

$$
V = q_1^{-5}(147q_1^6 + 441q_1^4q_2^2 + 56q_1^2q_2^4 + 4q_2^6). \tag{5.15}
$$

Second order variational equations for this potential possess logarithmic terms that suggests its non-integrability. But analysis of this case is not finished because equations on unknowns a and b obtained from condition of vanishing of all minors of $\boldsymbol{A}$ of degree $r+1$ are very complicated polynomials of a and b of degree 11. Thus, the problem of finding all their solutions is very hard. Potential (5.15) is just one example corresponding to the simplest solution of this system.

For $k = 3$ and $p \in \{-1, 0, 1\}$ system (5.14) does not possesses solution which gives a potential with required properties. For $p = 2$ obtained potentials of required degree $r = 5$

have very complicated coefficients and verification if they possess all required properties as well as their effective subsequent analysis seems to be impossible.

These examples show that the reconstruction of the potential from the given non-trivial eigenvalues of Hessians at all proper Darboux points is very hard task. Furthermore, if we are lucky and we made this reconstruction effectively, then we obtain potential that satisfies all known necessary integrability conditions but very often it is non-integrable.

5.4 Potentials without proper Darboux points

Another problem is related to the integrability analysis of rational potentials without proper Darboux points given in equation (4.28). For them we have no integrability obstructions. Thus at this moment the only tool that we have to our disposal is the direct method of search of first integrals [5]. Such analysis for this class of potentials was already done. Namely, if we make canonical transformation

$$z_1 = q_2 - \mathrm{i}q_1, \quad z_2 = q_2 + \mathrm{i}q_1, \qquad y_1 = \frac{1}{2}(p_2 + \mathrm{i}p_1), \quad y_2 = \frac{1}{2}(p_2 - \mathrm{i}p_1),$$

then Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + c(q_1 + \mathrm{i}q_2)^{\alpha}(q_1 - \mathrm{i}q_2)^{\beta}, \quad \alpha + \beta = k, \quad \alpha, \beta \in \mathbb{Z}, \qquad c \in \mathbb{C}^{\star}$$

takes the form

$$H = 2y_1y_2 + cz_1^{\alpha}z_2^{\beta}. \tag{5.16}$$

Integrability analysis for this class of Hamiltonian systems with not only integer but also with rational α and β was already made in [10] and many integrable rational potentials were found.

Acknowledgments

The authors are very grateful to Andrzej J. Maciejewski for many helpful comments and suggestions concerning improvements and simplifications of some results.

References

[1] D. Boucher. *Sur des équations différentielles linéares paramétrées, une application aux systémes hamiltoniens*. PhD thesis, Université de Limoges, France, 2000.

[2] G. Casale, G. Duval, A. J. Maciejewski, and M. Przybylska. Integrability of Hamiltonian systems with homogeneous potentials of degree zero. *Phys. Lett. A*, 374(3): 448–452, 2010.

[3] B. Dorizzi, B. Grammaticos, and A. Ramani. A new class of integrable systems. *J. Math. Phys.*, 24(9):2282–2288, 1983.

[4] G. Duval and A. J. Maciejewski. Jordan obstruction to the integrability of homogeneous potentials. *Ann. Inst. Fourier*, 59(7):2839–2890, 2009.

[5] J. Hietarinta. Direct methods for the search of the second invariant. *Phys. Rep.*, 147 (2):87–154, 1987.

[6] T. Kimura. On Riemann's equations which are solvable by quadratures. *Funkcial. Ekvac.*, 12:269–281, 1969/1970.

[7] J. J. Kovacic. An algorithm for solving second order linear homogeneous differential equations. *J. Symbolic Comput.*, 2(1):3–43, 1986.

[8] A. J. Maciejewski and M. Przybylska. All meromorphically integrable 2D Hamiltonian systems with homogeneous potentials of degree 3. *Phys. Lett. A*, 327(5-6): 461–473, 2004.

[9] A. J. Maciejewski and M. Przybylska. Darboux points and integrability of Hamiltonian systems with homogeneous polynomial potential. *J. Math. Phys.*, 46(6):062901, 33 pages, 2005.

[10] A. J. Maciejewski, M. Przybylska, and A.V. Tsiganov. On algebraic construction of certain integrable and super-integrable systems. *Physica D*, 240(18):1426–1448, 2011.

[11] J. J. Morales Ruiz. *Differential Galois theory and non-integrability of Hamiltonian systems*, volume 179 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[12] J. J. Morales-Ruiz and J.-P. Ramis. A note on the non-integrability of some Hamiltonian systems with a homogeneous potential. *Methods Appl. Anal.*, 8(1):113–120, 2001.

[13] M. Przybylska. Darboux points and integrability of homogenous Hamiltonian systems with three and more degrees of freedom. *Regul. Chaotic Dyn.*, 14(2):263–311, 2009.

[14] A. Ramani, B. Dorizzi, and B. Grammaticos. Painlevé conjecture revisited. *Phys. Rev. Lett.*, 49(21):1539–1541, 1982.

[15] Wolfram Research Inc. The Wolfram Functions Site: functions.wolfram.com. URL `http://functions.wolfram.com/Polynomials/JacobiP/`.

[16] M. Studziński. *Integrability studies of a certain class of homogeneous rational potentials*. B.Sc. thesis, N. Copernicus University, Toruń, Poland, 2009. in Polish.

[17] M. Studziński. *Integrability studies of homogeneous rational potentials*. M.Sc. thesis, N. Copernicus University, Toruń, Poland, 2010. in Polish.

[18] H. Yoshida. A criterion for the nonexistence of an additional integral in Hamiltonian systems with a homogeneous potential. *Phys. D*, 29(1-2):128–142, 1987.